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LMI-Based Control Design and Stability Analysis with a Conformable Fractional Derivative (CFD) for a Nonlinear Inverted Pendulum

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Index Terms

Inverted pendulum, conformable fractional derivative, LMI, nonlinear systems and Lyapunov stability.

Abstract

This paper uses a conformable fractional derivative (CFD) to analyze stability and design controls for nonlinear systems, specifically the inverted pendulum. The system is described as a linear component and a nonlinear component that satisfies a Lipschitz condition, and a nonlinear state-space model is generated. An adequate stability condition is obtained using a quadratic Lyapunov function, which is then converted into a linear matrix inequality (LMI) via the Schur complement, with an additional term that accounts for the worst-case effect of nonlinearities. A feedback gain that ensures stability is obtained by resolving this inequality. The Lipschitz constant determines the controller's aggressiveness. The performance of the controller is shown when the integer order mathematical model is replaced with fractional order one.

تصميم التحكم القائم على متباينات المصفوفات الخطية (LMI) وتحليل الاستقرار مع مشتق كسري مطابق (CFD) لبندول معكوس غير خطي

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الكلمات المفتاحية

البندول المعكوس، مشتق كسري مطابق، متباينة المصفوفة الخطية، أنظمة غير خطية واستقرار ليابونوف

الملخص

تستخدم هذه الورقة البحثية المشتقة الكسرية القابلة للتوافق (CFD) لتحليل الاستقرار وتصميم أنظمة التحكم للأنظمة غير الخطية، وبشكل خاص للبندول المعكوس. حيث يوصف النظام على أنه مكوّن خطي ومكوّن غير خطي يحقق شرط ليبشيتز(Lipschitz) ، ومن ثم يتم اشتقاق نموذج غير خطي في فضاء الحالة. يتم الحصول على شرط مناسب للاستقرار باستخدام دالة ليابونوف التربيعية، ثم يُحوَّل هذا الشرط إلى متباينة المصفوفة الخطية (LMI) باستخدام متممة شور(Schur Complement) ، مع إضافة حد يأخذ في الاعتبار أسوأ تأثير محتمل للحالة الغير خطية. ويتم الحصول على كسب تغذية راجعة يضمن الاستقرار من خلال حل هذه المتباينة. كما يحدد ثابت ليبشيتز مدى شدة أو قوة أداء المتحكم. ويتم توضيح أداء المتحكم عند استبدال النموذج الرياضي ذي الرتبة الصحيحة بنموذج ذي رتبة كسرية.

1. INTRODUCTION

Since it is fundamentally unstable, highly nonlinear, and underactuated, the inverted pendulum system is one of the most traditional and difficult benchmark problems in control engineering. This makes it a perfect testbed for new control strategies before implementing them in real-world systems like humanoid robots, autonomous vehicles, and rockets during launch [1],[2].

Although PID and other traditional control methods are easy to use, they frequently have trouble with big disturbances and nonlinearities [3]. More sophisticated

techniques have performed better. However, they can be computationally intensive and usually require exact mathematical representations [4]. The majority of current control schemes make the assumption that system dynamics are governed by integer-order differential equations, yet many physical events display fractional-order behavior that is difficult for standard models to adequately represent.[5]. The conformable fractional derivative (CFD), which satisfies standard properties like the chain rule, product rule, and derivative of a constant, was proposed in 2014 by Khalil and his colleagues, including Al Horani and Yousef. It reduces exactly to the ordinary derivative when the fractional order equals one [1], [5].

For more than a century, Lyapunov's direct method which commonly uses a quadratic Lyapunov function to generate stability conditions has been the cornerstone of stability analysis for control design, where a basic consideration is whether the system will remain stable.[6]. These conditions frequently result in matrix inequalities that can be expressed as linear matrix inequalities (LIMs), which are convex optimization problems that can be effectively resolved with interior-point techniques and software.[7]. The stability requirement for the linearized inverted pendulum can be converted into a straightforward linear matrix inequality (LMI).[8].

The system's nonlinearities, such as the sine and cosine of the pendulum angle and the square of the angular velocity, present the true difficulty, though, as they cannot be disregarded when the pendulum deviates significantly from the vertical position. [9]. The nonlinear function satisfies a Lipschitz condition under plausible assumptions, enabling us to take nonlinearities into account. [4]. An improved stability condition can be obtained by incorporating the Lipschitz condition (LC) and handling the cross terms that arise in the Lyapunov derivative using Young's inequality.[4]. The Schur complement lemma can be utilized to linearize this new condition, producing a larger linear matrix inequality (LMI) that ensures stability even when nonlinear terms are present. [10], [11], [12], [13].

Three key concepts are included in this paper: linear matrix inequality techniques (LMI) for systematic controller design, Lyapunov stability theory for rigorous analysis, and conformable fractional derivative (CFD) for flexible modelling [10]. The main contributions include creating a comprehensive nonlinear state-space model for the inverted pendulum using the conformable fractional derivative (CFD), using a quadratic Lyapunov function to derive sufficient conditions for asymptotic stability, extending the analysis to account for nonlinearities using Young's inequality and the Lipschitz condition (LC), using MATLAB to solve the linear matrix inequalities (LMIs) numerically to obtain explicit controller gains, and comparing the performance of the linear controller with the nonlinear one [14].

The structure of the paper is as follows: The mathematical preliminaries are covered in Section two. The inverted pendulum's nonlinear state-space model is derived in Section three. The primary theoretical findings, specifically LMI-based stability requirements and the construction of a state feedback controller with proof, are presented in Section four. The simulation results of the considered system are shown in section 5 With simulation results, and the conclusion ins in section 6.

2. PRELIMINARIES AND PROBLEM FORMULATION

The basic definitions and lemmas utilized in this study are presented in this part, along with significant details about conformable fractional derivatives (CFD). The following definition applies to the conformable fractional derivatives used:

Definition 2.1: **(Conformable Fractional Derivative)** [2], [10]**:** For a function $f:[0,+\infty)\to\mathbb{R}$, the conformable fractional derivative of order $\alpha\in(0,1]$ is defined as:

$$D_t^{\alpha}f(t)=\lim_{\epsilon\to 0}\frac{f(t+\epsilon t^{1-\alpha})-f(t)}{\epsilon} \qquad (1)$$

$\forall t>0$ and $0<\alpha\leq 1$.

Property 1 [1]**:** If f is differentiable, then (1) is equivalent to:

$$D_t^{\alpha}f(t)=t^{1-\alpha}\ \frac{df(t)}{dt} \qquad (2)$$

For simulation purposes, this property is essential since it enables us to convert the conformable differential equations into ordinary differential equations.

Lemma 2.1: [10], [11], [15]: Let $x:[a,+\infty)\to\mathbb{R}^n$ such that $D_t^{\alpha}x(t)$ is well defined on $[a,+\infty)$. Consider an SDP matrix P. So, $D_t^{\alpha}x^TPx(t)$ $\exists$ on $(a,+\infty)$ and we have, $\forall t>a$:

$$D_t^{\alpha}x^TPx(t)=2x^T(t)PD_t^{\alpha}x(t),$$

Definition 2.2: [16]**.** A function $\Delta:\mathbb{R}^n\to\mathbb{R}^n$ is said to satisfy the Lipschitz condition in a domain $D\subset\mathbb{R}^n$ if there exists a constant $L>0$ such that:

$$\|\Delta(x_1)-\Delta(x_2)\|\leq L\|x_1-x_2\|,\quad \forall x_1,x_2\in D \qquad (3)$$

The constant L is called the **Lipschitz constant**.

Corollary 1 [3]: If $\Delta(0)=0$, then (3) implies:

$$\|\Delta(x)\|\leq L\|x\|,\quad \forall x\in D$$

Equivalently:

$$\Delta(x)^T\Delta(x)\leq L^2x^Tx=L^2x^TI_nx$$

Remark 1 [17]**:** The Lipschitz condition is widely used in nonlinear control theory to quantify the "growth rate" of nonlinear functions. Many physical systems, including the inverted pendulum with bounded angles and velocities, satisfy this condition locally.

Definition 2.3 (Asymptotic Stability) [18]**:** The equilibrium point $x_0=0$ is said to be asymptotically stable if:

- For every $\varepsilon>0$, there exists $\delta>0$ such that $\|x(0)\|<\delta$ implies $\|x(t)\|<\varepsilon$ for all $t>0$ (stability in the sense of Lyapunov).
- There exists $\eta>0$ such that $\|x(0)\|<\eta$ implies

$$\lim_{t\to\infty} \| x(t) \| = 0$$

Theorem 1 (Lyapunov's Direct Method for Conformable Fractional Systems) [19]**:** Consider the conformable fractional-order system:

$$D_t^\alpha x = f(x), \qquad f(0) = 0$$

with $\alpha \in (0,1]$. Let $V:\mathbb{R}^n \to R$ be a continuously differentiable function. If there exists a positive definite function $V(\boldsymbol{x})$ such that:

$$D_t^\alpha V(x) < 0, \qquad \forall x \neq 0$$

then the equilibrium point is **asymptotically stable**.

Remark 2 [10], [11], [12]**:** The most common choice for $V(x)$ is the quadratic Lyapunov function:

$$V(x) = x^T P x$$

Where $P = P^T > 0$ is a symmetric positive definite matrix

Lemma 2.2 (Schur Complement) [10], [13]: Consider a symmetric matrix partitioned into blocks:

$$M = \begin{bmatrix} Q & S \\ S^T & -R \end{bmatrix}$$

$$M < 0 \quad \Leftrightarrow Q + SR^{-1}S^T < 0$$

where:

- $Q \in \mathbb{R}^{n\times n}$, $R \in \mathbb{R}^{m\times m}$ are symmetric and $S \in \mathbb{R}^{n\times m}$
- If $R > 0$, then the following equivalence holds:

$$M < 0 \quad \Leftrightarrow Q - SR^{-1}S^T < 0$$

3. STATE-SPACE MODEL OF THE INVERTED PENDULUM

Definition 3.1 (Inverted Pendulum System) [20]**:**

A benchmark system for classical nonlinear control is the inverted pendulum on a cart. It is made up of: A mass $\boldsymbol{M}$ cart is travelling horizontally along a track. A pivot connects a pendulum with mass $\boldsymbol{m}$ and length $\boldsymbol{l}$ to the cart. A horizontal control force $\boldsymbol{u}$ is applied to the cart. The cart motion is opposed by the viscous friction coefficient $\boldsymbol{b}$.

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} = \begin{bmatrix} x(t) \\ \dot{x}(t) \\ \theta(t) \\ \dot{\theta}(t) \end{bmatrix} \in R^4$$

The nonlinear state-space representation is:

$$D_t^\alpha x_1 = x_2$$

$$D_t^\alpha x_2 = \frac{1}{M + m\,sin^2 x_3}[u - bx_2 + m \sin x_3 (l\, x_4^2 + g\, cos\, x_3)]$$

$$D_t^\alpha x_3 = x_4$$

$$D_t^\alpha x_4 = \frac{1}{l\,(M + m\,sin^2 x_3)}[(u - bx_2)\, cos x_3 + (M+m) g \sin x_3 - mlx_4^2 \sin x_3 \cos x_3]$$

Around the equilibrium point

$$x_0 = [0,0,0,0]^T$$

the linearized matrices are:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -\frac{b}{M} & -\frac{mg}{M} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -\frac{b}{Ml} & \frac{(M+m)g}{Ml} & 0 \end{bmatrix}, B = \begin{bmatrix} 0 \\ \frac{1}{M} \\ 0 \\ \frac{1}{Ml} \end{bmatrix} \quad (4)$$

The linearized conformable fractional-order system is:

$$D_t^\alpha x = Ax + Bu \quad (5)$$

4. STATE FEEDBACK CONTROLLER DESIGN

In this section, the following nonlinear CFOS is given:

$$\begin{cases} D_t^\alpha x(t) = Ax(t) + Bu(t) + \Delta(x), \\ y(t) = Cx(t), \end{cases} \quad (6)$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$ and $y(t) \in \mathbb{R}^p$. $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}, and\ C \in \mathbb{R}^{p\times n}$ is a given that they are known constant matrices. Additionally, it is assumed that (A, B) is stabilizable.

The state feedback control law, considered in this paper, is:

$$u(t) = -Kx(t) \quad (7)$$

Where $K \in R^{1\times 4}$ is the feedback gain matrix to be determined.

The following theorem shows the main result:

Theorem 2. One considers the system (6). Under conditions (3), the control law (7) stabilizes system (6) if $\exists\ X = X^T > 0$, W and $\varepsilon > 0$, such that the following LMI holds:

$$\begin{bmatrix} AX + XA^T - BY - Y^T B^T + \varepsilon L^2 X^2 & X \\ X & -\varepsilon I \end{bmatrix} < 0$$

with:

$$X = X^T > 0\,, X \in R^{4\times 4}\,, Y \in R^{1\times 4}$$

Proof.

By employing (6) and (7), along with **Lemma 2.1**, and substituting $\Delta(x)$ with Δ, one obtains:

$$D_t^\alpha V(x) = (D_t^\alpha x)^T P x + x^T P (D_t^\alpha x) \quad (8)$$

Substituting (6) into (7):

$$D_t^\alpha x = Ax - BKx = (A - BK)x \quad (9)$$

Substitute (7) into the original nonlinear system:

$$D_t^\alpha x = f(x, -Kx) \quad (10)$$

We can write:

$$D_t^\alpha x = (A - BK)x + [f(x, -Kx) - (A - BK)x] \quad (11)$$

Define the nonlinear function:

$$\Delta(x) = f(x, -Kx) - (A - BK)x \quad (12)$$

Thus:

$$D_t^\alpha x = (A - BK)x + \Delta(x) \quad (13)$$

Explicit Expression of $\Delta(x)$

$$\Delta(x) = \begin{bmatrix} 0 \\ \Delta_2(x) \\ 0 \\ \Delta_4(x) \end{bmatrix} \quad (14)$$

According to (3) and $\Delta(0) = 0$ (because at equilibrium, the nonlinear part vanishes), we have:

$$\| \Delta(x) \| \leq L \| x \|, \forall x \in D \quad (15)$$

Equivalently:

$$\Delta(x)^T \Delta(x) \leq L^2 x^T x = L^2 x^T I\, x \quad (16)$$

Choose the quadratic Lyapunov function and using the chain rule for the conformable fractional derivative same as (8) then substitute $D_t^\alpha x$ from (13):

$$D_t^\alpha V = [(A - BK)x + \Delta]^T Px + x^T P[(A - BK)x + \Delta] \quad (17)$$

$$D_t^\alpha V = x^T (A - BK)^T Px + \Delta^T Px + x^T P(A - BK)x + x^T P\Delta \quad (18)$$

Note that $\Delta^T Px$ and $x^T P\Delta$ are scalars, and they are equal because:

$$\Delta^T Px = (\Delta^T Px)^T = x^T P^T \Delta = x^T P\Delta$$

(Since $P = P^T$)

Therefore:

$$D_t^\alpha V = x^T[(A - BK)^T P + P(A - BK)]x + 2x^T P\Delta \quad (19)$$

For any $\varepsilon > 0$, the following inequality holds[14]:

$$2x^T P\Delta \leq \frac{1}{\varepsilon} x^T P^2 x + \varepsilon \Delta^T \Delta \quad (20)$$

Since $x^T P\Delta = \Delta^T Px$, we obtain (20). And apply the Lipschitz Condition (16) and into (20):

$$2x^T P\Delta \leq \frac{1}{\varepsilon} x^T P^2 x + \varepsilon L^2 x^T x \quad (21)$$

Upper Bound for $D_t^\alpha V$

Substitute (21) into (19):

$$D_t^\alpha V \leq x^T[(A - BK)^T P + P(A - BK)]x + \frac{1}{\varepsilon} x^T P^2 x + \varepsilon L^2 x^T x \quad (22)$$

Factor x^T on the left and x on the right:

$$D_t^\alpha V \leq x^T \left[(A - BK)^T P + P(A - BK) + \frac{1}{\varepsilon} P^2 + \varepsilon L^2 I\right] x \quad (23)$$

Asymptotic Stability Condition

For asymptotic stability, we require $D_t^\alpha V < 0$ for all $x \neq 0$. A sufficient condition is that the matrix in brackets is negative definite:

$$(A - BK)^T P + P(A - BK) + \frac{1}{\varepsilon} P^2 + \varepsilon L^2 I < 0 \quad (24)$$

This is a nonlinear matrix inequality (because of the $\frac{1}{\varepsilon} P^2$ term).

Apply Schur Complement to Linearize $\frac{1}{\varepsilon} P^2$

Let us define:

$$Q = (A - BK)^T P + P(A - BK) + \varepsilon L^2 I \quad (25)$$

$S = P,\ R = \varepsilon I$ *(note:* $R > 0$ *since* $\varepsilon > 0$*)*

Compute:

$$Q + SR^{-1}S^T = Q + P(\varepsilon I)^{-1} P^T = Q + \frac{1}{\varepsilon} P^2 \quad (26)$$

Therefore, condition (24) $Q + \frac{1}{\varepsilon} P^2 < 0$ is equivalent to:

$$\begin{bmatrix} (A - BK)^T P + P(A - BK) + \varepsilon L^2 I & P \\ P & -\varepsilon I \end{bmatrix} > 0 \quad (27)$$

This is a linear matrix inequality in P (since K appears linearly as well).

Change of Variables to Obtain Standard LMI Form

To define the new variable: $X = P^{-1} \in R^{4\times4}$, $X = X^T > 0$ and $Y = KX \in R^{1\times4}$

Consider the block diagonal matrix:

$$\begin{bmatrix} X & 0 \\ 0 & I \end{bmatrix} \in R \quad (28)$$

Pre-multiply and post-multiply (27) by this matrix:

$$\begin{bmatrix} X & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} (A - BK)^T P + P(A - BK) + \varepsilon L^2 I & P \\ P & -\varepsilon I \end{bmatrix} \begin{bmatrix} X & 0 \\ 0 & I \end{bmatrix} < 0 \quad (29)$$

Simplify Each Block Upper-left block**:**

$$X(A - BK)^T PX + XP(A - BK)X$$

Expand:

$$= X(A - BK)^T PX + XP(A - BK)X + \varepsilon L^2 X^2 \quad (30)$$

Now:

$$X(A - BK)^T PX = X(A^T - K^T B^T)(PX) = (XA^T - XK^T B^T)I$$

Since

$$PX = I = XA^T - XK^T B^T$$

But

$$XK^T = (KX)^T = Y^T XK^T$$
$$= (KX)^T = Y^T$$

Thus:

$$X(A - BK)^T PX = XA^T - Y^T B^T \quad (31)$$

Similarly:

$$XP(A - BK)X = (A - BK)X$$
$$= AX - BKX = AX - BY \quad (32)$$

Therefore, the upper-left block becomes:

$$XA^T - Y^T B^T + AX - BY + \varepsilon L^2 X^2 = AX + XA^T - BY - Y^T B^T + \varepsilon L^2 X^2 \quad (33)$$

Upper-right block: $XP = I$ and **Lower-left block:** $PX = I$

Lower-right block remains: $-\varepsilon I$

Final LMI Formulation by Combining all blocks:

$$\begin{bmatrix} AX + XA^T - BY - Y^T B^T + \varepsilon L^2 X^2 & X \\ X & -\varepsilon I \end{bmatrix} < 0 \quad (34)$$

with:

$$X = X^T > 0 \quad , X \in R \quad Y \in R^{1\times 4} \quad \varepsilon > 0$$

Once (34) is solved for X and Y, the feedback gain matrix is recovered as:

$$K = YX^{-1} \in R^{1\times 4} \quad (35)$$

Theorem 3: (LMI-Based Stability Condition in the Presence of Nonlinearities)

Consider the Conformable Fractional Derivative (CFD) inverted pendulum system described by (13), where the nonlinear function $\Delta(x)$ satisfies the Lipschitz condition (3) with constant $L > 0$ in the operating domain $D \subset R^4$. If there exists a symmetric positive definite matrix $X \in R^{4\times 4}$, a matrix $Y \in R^{1\times 4}$, and a scalar $\varepsilon > 0$ such that the LMI (34) holds, then the closed-loop system (13) with the state-feedback controller $u = -Kx$, where $K = YX^{-1}$, is **asymptotically stable**.

This ends the proof.

5. SIMULATION RESULTS

This section presents a simulation analysis and a numerical example to demonstrate the effectiveness of the suggested stabilizing approach for the under consideration fractional order nonlinear systems. We can simulate using MATLAB's code function since the Confirmable Fractional Derivative (CFD) specification includes the integer order derivative. Equation (6) employs the system parameters, which are listed in table 1.

Table 1: The parameters of the inverted pendulum system

Parameter	Symbol	Value-Unit	Description
Cart mass	M	1 kg	Mass of the cart
Pendulum mass	m	0.1 kg	Mass of the pendulum
Pendulum length	l	0.5 m	Half-length of pendulum (pivot to center of mass)
Gravity	g	9.81m/s^2	Gravitational acceleration
Friction coefficient	b	0.1 $N.s/m$	Viscous friction of the cart

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -0.1 & -0.981 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -0.2 & 21.582 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 2 \end{bmatrix}$$

$$, c = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \Delta(x) = \begin{bmatrix} 0 \\ \Delta_2(x) \\ 0 \\ \Delta_4(x) \end{bmatrix},$$

$$\Delta_2(x) = \frac{1}{M + m\ sin^2 x_3}[-Kx - bx_2 + m \sin x_3 (lx_4^2 + g \cos x_3)] - \left(-\frac{1}{M}Kx - \frac{b}{M}x_2 - \frac{mg}{M}x_3\right)$$

$$\Delta_4(x) = \frac{1}{l(M + m\ sin^2 x_3)}[(-Kx - bx_2)cos\ x_3 + (M + m)gsin\ x_3 - mlx_4^2 \sin x_3 \cos x_3] - (-\frac{b}{Ml}x_2 + \frac{(M + m)g}{Ml}x_3 - \frac{1}{Ml}Kx)$$

Conditions (3) have been met with $L = 2.34$. The LMI (34) is solved with $\varepsilon = 0.1$ using the MATLAB (LMI) toolbox. The result is as follows:

$$X = \begin{bmatrix} 0.0085 & 0.0048 & -0.0012 & 0.0003 \\ 0.0048 & 0.0129 & -0.0007 & 0.0025 \\ -0.0012 & -0.0007 & 0.0190 & 0.0037 \\ 0.0003 & 0.0025 & 0.0037 & 0.0251 \end{bmatrix}$$

$$Y = [-0.1165 \quad -0.1867 \quad 2.4205 \quad 0.4932],$$

$$K = YX^{-1} = [-13.7098 \quad -14.3132 \quad 125.6020 \quad 19.3570]$$

$$(A - BK) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 13.7098 & 14.3132 & -126.5830 & -19.3570 \\ 0 & 0 & 0 & 1 \\ 27.4196 & 28.4264 & -229.6220 & -38.7140 \end{bmatrix}$$

The simulation results confirm that the LMI-based controller successfully stabilizes the nonlinear conformable fractional-order inverted pendulum system with $\alpha = 0.75$. As shown in Figure 1, the cart position stays within ± 0.025 m with no oscillations and returns smoothly to equilibrium after an initial peak at $t = 1.5$s, see figure (1-a). The control input is smooth and completely free of chattering, making it suitable for real actuators figure (1-b). The pendulum angle, with 10° initial condition, settles in about 2.5 seconds with only $2 - 3\%$ overshoot and remains within 0.2 of upright, showing well-damped behavior without visible oscillations figure (1-c), while both cart and angular

velocities converge smoothly to zero without sustained oscillations figure (1-d). For the sake of comparison, the same controller is compared to an integer order controller, setting ($\alpha = 1$). The response of the system with both controllers are shown in figure 2. The fractional-order system settles faster with less overshoot, demonstrating beneficial damping from the conformable derivative, whereas the integer-order response shows more oscillations between $1-3$ seconds (2-a). Additionally, the fractional-order controller achieves better performance with lower control effort, indicating more efficient energy usage (2-b). Finally, the phase portrait in Figure 3 shows the trajectory spiraling smoothly toward the equilibrium at $(0,0)$ without limit cycles, confirming asymptotic stability and robust performance. Overall, the LMI-based controller provides excellent stabilization for the fractional-order inverted pendulum, with faster response, lower overshoot, smoother control, and better energy efficiency than the integer-order equivalent.

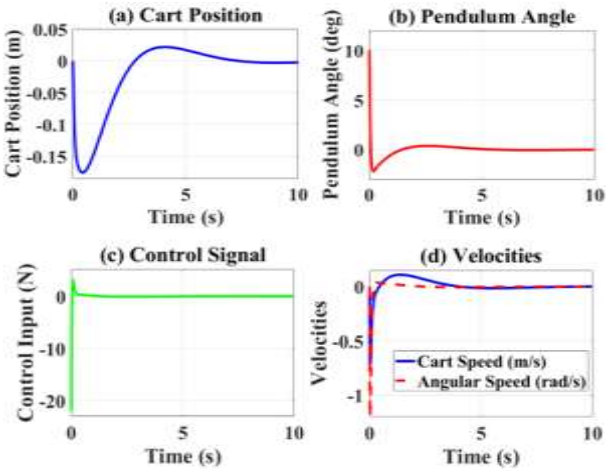


Fig 1: Conformable Fractional-Order Inverted Pendulum Response ($\alpha = 0.75$)

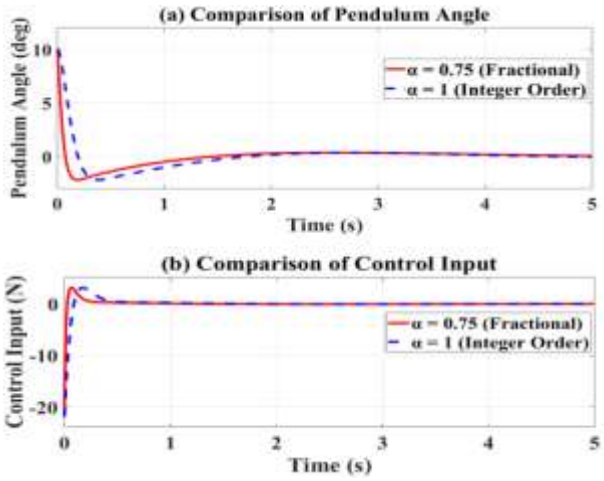


Fig 2: Performance Comparison: Fractional vs Integer Order

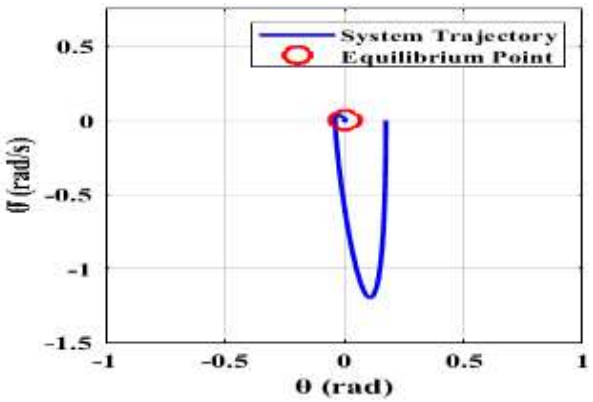


Fig 3: Phase Portrait of Pendulum

6. CONCLUSIONS

This study successfully demonstrated the stabilization of a conformable fractional-order inverted pendulum using an LMI-based state feedback controller. The simulation results clearly show that the fractional-order system ($\alpha = 0.75$) outperforms the integer-order system ($\alpha = 1$). The smooth control signals and exponential decay of the state norm confirm the effectiveness and stability of the proposed LMI approach. These findings validate that fractional-order modeling offers significant advantages for underactuated mechanical systems, providing improved transient response and reduced control effort compared to traditional integer-order control.

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