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Fekete-Szegö Inequality For Certain Subclass Of Multivalent Functions Associated with Al-Oboudi Differential Operator

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Abstract
In this paper, we define and study a new class $H^{j}_{\delta,p,b}(\varphi)$ of multivalent analytic functions defined on the open unit disk $\mathcal{U}$, where this class is defined by means of the Al-Aboudi differential operator. Given the importance of coefficient bounds in shaping the behavior of multivalent functions, the Fekete–Szegő inequality is used to derive the coefficient bounds for functions belonging to this class, in order to illustrate the role of the Al-Aboudi differential operator in influencing this behavior. This study also presents a unified framework that combines the Al-Aboudi differential operator with some classical results on coefficient bounds, leading to generalizations that serve as a foundation for subsequent developments in the theory of geometric functions.

متباينة فيكيت - شيقو لفصل من الدوال متعددة التكافؤ و المرتبط بمؤثر العبودي التفاضلي

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الكلمات المفتاحية
دوال متعددة التكافؤ ، الدوال النجمية ، الرتبة الأسية ، متباينة فيكيت – شيغو، مؤثر العبودي التفاضلي

الملخص
سنقوم في هذا البحث بتعريف و دراسة فصل جديد $H^{j}_{\delta,p,b}(\varphi)$ من الدوال التحليلية متعددة التكافؤ المعرفة على قرص الوحدة المفتوح $\mathcal{U}$، حيث تم تعريف هذا الفصل بواسطة مؤثر العبودي التفاضلي، ونظراً لما لحدود المعاملات لهذه الدوال من أهمية في تشكيل سلوك الدوال متعددة التكافؤ فقد تم استخدام متباينة فيكيت - شيقو في استنتاج حدود المعاملات للدوال المنتمية لهذا الفصل لتوضيح دور مؤثر العبودي التفاضلي في التأثير على هذا السلوك . كما تقدم هذه الدراسة إطار موحد يجمع بين مؤثر العبودي التفاضلي و بعض النتائج الكلاسيكية لحدود المعاملات مما يؤدي إلى تعميمات تكون بمثابة الأساس لتطورات لاحقة في نظرية الدوال الهندسية .

1. INTRODUCTION AND DEFINITIONS

Let $\mathcal{A}$ denote the class of functions $f(z)$ is normalized by :

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n ; \quad z \in \mathcal{U} \qquad (1.1)$$

which are analytic in the open unit disk

$\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$.

For $f(z) \in \mathcal{A}$, Al-Oboudi [2] introduced and investigated the following operator:

$$D^0_\delta f(z) = f(z)$$

$$D^1_\delta f(z) = (1-\delta) f(z) + \delta z f'(z)$$

$$D^j_\delta f(z) = D^1_\delta \left(D^{j-1}_\delta f(z) \right) , \quad j \in \mathbb{N} \qquad (1.2)$$

If $f(z)$ is given by the series above, then

$$D^j_\delta f(z) = z + \sum_{n=2}^{\infty} [1 + (n-1)\delta]^j a_n z^n ,$$

$$j \in \mathbb{N}_0 = \mathbb{N} \cup \{0\} \qquad (1.3)$$

Note that, when $\delta = 1$, we obtain Sălăgean operator [10]

Let $\mathcal{A}(p)$ denote the class of functions $f(z)$ defined by

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n} \quad , p \in \mathbb{N} \qquad (1.4)$$

Which are analytic and p-valent in the disk $\mathcal{U}$.

For $f(z) \in \mathcal{A}(p)$, we define the generalized Al-Oboudi operator $D^j_{\delta,p}$ as follows:

$$D^0_{\delta,p} f(z) = f(z)$$

$$D^1_{\delta,p} f(z) = \frac{\delta}{p} z f'(z) + \frac{1}{p}(1-\delta) f(z) + \frac{p-1}{p}(1-\delta) z^p$$

and

$$D^j_{\delta,p} f(z) = D^1_{\delta,p} \left(D^{j-1}_{\delta,p} f(z) \right) , \quad j \in \mathbb{N} \qquad (1.5)$$

From this definition, we can verify that

$$D_{\delta,p}^{j} f(z) = z^p + \sum_{n=1}^{\infty} \left[\frac{1+(p+n-1)\delta}{p}\right]^j a_{p+n}\, z^{p+n}$$

$$= z^p + \sum_{n=1}^{\infty} G_n(\delta,p)\, a_{p+n}\, z^{p+n} \qquad (1.6)$$

where

$$G_n(\delta,p) = \left[\frac{1+(p+n-1)\delta}{p}\right]^j \qquad (1.7)$$

Remark 1.1

- If $p = 1$, then $D_{\delta,p}^{j} f(z) = D_{\delta}^{j} f(z)$, the operator studied by Al-Oboudi [2].
- If $p = 1$, $\delta = 1$, we obtain the Sălăgean operator [9].

1.1. Starlike and convex functions of complex order

Definition 1.1 A function $f(z) \in \mathcal{A}(p)$ is called p-valent starlike of order α (denoted by $S_p^*(\alpha)$) if $f(z)$ satisfies the conditions:

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha, \qquad 0 \le \alpha < p, z \in \mathcal{U} \qquad (1.8)$$

and

$$\int_0^{2\pi} \mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} d\theta = 2p\pi \qquad (1.9)$$

Also, a function $f(z) \in \mathcal{A}(p)$ is called p-valent convex of order α (denoted by $C_p(\alpha)$)if

$$\mathrm{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\} > \alpha\,, \quad 0 \le \alpha < p, z \in \mathcal{U} \qquad (1.10)$$

and

$$\int_0^{2\pi} \mathrm{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\} d\theta = 2p\pi \qquad (1.11)$$

For $0 \le \alpha < p,\ p \in \mathbb{N}$, and $z \in \mathcal{U}$.

The class $S_p^*(\alpha)$ was introduced by Patil and Thakare [7], and the class $C_p(\alpha)$was introduced by Owa [6]. For $p = 1$, the classes $S_1^*(\alpha) = S^*(\alpha)$ and $C_1(\alpha) = C(\alpha)$ were studied by Silverman [12].

From (1.8) and (1.10) we note that :

$$f(z) \in C_p(\alpha) \Leftrightarrow \frac{zf'(z)}{p} \in S_p^*(\alpha) \qquad (1.12)$$

For $0 \le \alpha < p$.

Example 1.2. Let the function $f: \mathcal{U} \to \mathbb{C}$, be defined by

$$f(z) = \frac{z^p}{(1-z)^{2(p-\alpha)}}$$

where : $0 \le \alpha < p$, $p \in \mathbb{N}$, $z \in \mathcal{U}$.

then $f(z) \in S_p^*(\alpha)$.

Now , define $f(z)$ by its derivative as :

$$f'(z) = pz^{p-1}(1-z)^{-2(p-\alpha)}$$

where : $0 \le \alpha < p$, $p \in \mathbb{N}$, $z \in \mathcal{U}$.

then $f(z) \in C_p(\alpha)$.

Definition 1.3. Let the functions $f(z)$ and $g(z)$ be analytic in $\mathcal{U}$,We say that the function $f(z)$ is subordinate to $g(z)$ in $\mathcal{U}$ (denoted by $f(z) \prec g(z)$) if there exists a Schwarz function $w(z)$ analytic in $\mathcal{U}$ with $w(0) = 0$ and $|w(z)| < 1$, such that

$$f(z) = g\big(w(z)\big)\,, \qquad z \in \mathcal{U}$$

Furthermore, if the function $g(z)$is univalent in $\mathcal{U}$ then we have

$$f(z) \prec g(z) \iff f(0) = g(0)$$

Define the function $\varphi(z)$ to be analytic function with positive real part and satisfies

$$\varphi(0) = 1\ ,\ \varphi'(z) > 0 \qquad (1.13)$$

which maps $\mathcal{U}$ onto a starlike region with respect to 1 and symmetric to the real axis.

Example 1.4. Consider the functions $f(z), g(z)$ analytic in $\mathcal{U}$, defined by :

$$f(z) = \frac{1+z^2}{1-z^2} \quad , \quad g(z) = \frac{1+z}{1-z}$$

then , $f(z) \prec g(z)$ and $w(z) = z^2$.

Definition 1.5. A function $f(z) \in \mathcal{A}(p)$ is called a starlike function of a complex order (denoted by $S_{b,p}^*(\varphi)$) if and only if

$$1+\frac{1}{b}\left\{\frac{1}{p}\frac{zf'(z)}{f(z)} - 1\right\} \prec \varphi(z) \qquad (1.14)$$

$$z \in \mathcal{U}\,, b \in \mathbb{C}^* = \mathbb{C} - \{0\}$$

A function $f(z) \in A(p)$ is called a convex function of a complex order (denoted by $C_{b,p}(\varphi)$) if and only if

$$1+\frac{1}{b}+\frac{1}{bp}\left\{1+\frac{zf''(z)}{f'(z)}\right\} \prec \varphi(z) \qquad (1.15)$$

$$z \in \mathcal{U}\,, b \in \mathbb{C}^*$$

The classes $S_{b,p}^*(\varphi)$ and $C_{b,p}(\varphi)$ were introduced and studied by Ali et. Al. [1]

Note that: $S^*_{b,1}(\varphi) = S^*_b(\varphi)$ and $C_{b,1}(\varphi) = C_b(\varphi)$, The classes $S^*_b(\varphi)$ and $C_b(\varphi)$ were studied by Ravichandran et. Al. [9]

It is clear that $S^*_{1,1}(\varphi) = S^*(\varphi)$ and $C_{1,1}(\varphi) = C(\varphi)$ The classes $S^*(\varphi)$ and $C(\varphi)$ were introduced and studied by Ma and Minda [5]

Example 1.6. Consider the analytic multivalent function

$$f(z) = z^p\,(1-z)^{-2\,b\,p}$$

where : $b \in \mathbb{C}^*$, $p \in \mathbb{N}$, $z \in \mathcal{U}$, and $\varphi(z) = \frac{1+z}{1-z}$

then $f(z) \in S^*_{b,p}(\varphi)$.

Now, let $f(z)$ be defined via its first derivative as follows:

$$f'(z) = p\,z^{p-1}(1-z)^{-2bp}$$

where : $b \in \mathbb{C}^*$, $p \in \mathbb{N}$, $z \in \mathcal{U}$, and $\varphi(z) = \frac{1+z}{1-z}$

then $f(z) \in C_{b,p}(\varphi)$.

1.2. The main subclass $H^{j}_{\delta,p,b}(\varphi)$

Definition 1.7. For $b \in \mathbb{C}^*$, $0 \le \beta \le 1$, $j \in \mathbb{N}_0, p \in \mathbb{N}$ we define the subclass of $\mathcal{A}(p)$ denoted by $H^{j}_{\delta,p,b}(\varphi)$ that consisting of functions $f(z)$ of form (1.4) and $\varphi(z)$ as in (1.13) and satisfying the following condition :

$$1+\frac{1}{b}\left\{\frac{1}{p}\frac{z\left(D^{j}_{\delta,p}f(z)\right)' + \beta z^2\left(D^{j}_{\delta,p}f(z)\right)''}{(1-\beta)\left(D^{j}_{\delta,p}f(z)\right) + \beta z\left(D^{j}_{\delta,p}f(z)\right)'} - 1\right\} \prec \varphi(z) \qquad (1.16)$$

We note that for suitable choices of δ, β, p, b we obtain the following subclasses:

1- For $\beta = 0$, $j = 0$ and $\delta = 1$, we get the class $H^{0}_{1,p,b}(\varphi) = S^*_{b,p}(\varphi)$ [1] .
2- For $\beta = 0$, $j = 0$, $\delta = 1$ and $p = 1$, we get the class $S^*_b(\varphi)$ [9].
3- For $\beta = 1$, $j = 0$, $\delta = 1$ and $p = 1$,we get the class $C_b(\varphi)$ [9] .
4- For $\beta = 0$, $j = 0$, $p = 1, \delta = 1$ and $b = 1$, we get the class $S^*(\varphi)$ [5] .
5- For $\beta = 1$, $j = 0$, $p = 1$, $\delta = 1$ and $b = 1$ we get the class $C(\varphi)$ [5].
6- For $\beta = 0$ and $\delta = 1$, we get the class $\mathcal{C}(\lambda, b, p; \phi)$ [8].

Over the last few years, there has been a noticeable shift toward investigating coefficient problems - specifically the Fekete-Szegő functional - for subclasses of analytic functions that incorporate the Al-Oboudi differential operator. Much of this recent work builds on the principle of subordination and the properties of the Schwarz function $w(z)$, often extending these concepts to include functions of complex order b . By applying well-established lemmas to handle the coefficients of functions with a positive real part , these studies aim to determine sharp upper bounds for various geometric classes [1] [3] [4] [8] [9] [14]. Following this line of research, the present paper introduces the class $H^{j}_{\delta,p,b}(\varphi)$ to provide more generalized framework that connects several previously studied cases.

1.3. Background and Related Work

Research on coefficient estimates and extremal problems has long been an important part of geometric function theory. The study of the Fekete–Szegö functional, attracted wide attention after its introduction for univalent functions. Later, Ma and Minda [5] introduced a unified approach to handle various subclasses of starlike and convex functions using differential subordination. Shanmugam and Sivasubramanian [11] then extended this direction by looking at the Fekete–Szegö inequality for classes involving combination of starlike and convex behaviors under subordination. This framework was further broadened to complex order b and multivalent (p-valent) functions by Ravichandran et al. [9] and Ali et al. [1].

Many authors worked on differential operators to study coefficient problems. Sălăgean [10] proposed a basic differential operator, which Al-Oboudi [2] later generalized with a parameter δ . Using this operator, Thirucheran and Stalin [13, 14] obtained Fekete–Szegö bounds for some analytic function classes. More recent work includes Aouf et al. [3] , who studied a q-analogue of Al-Oboudi's operator in meromorphic functions.

However, existing studies usually tackle these elements separately.

For example, Shanmugam and Sivasubramanian [11] focused on subordination, while Thirucheran and Stalin [13, 14] considered the Al-Oboudi operator mainly for univalent functions ($p = 1$), without including complex order or the combination parameter β. There is still a lack of a single framework that combines the generalized Al-Oboudi operator for p-valent functions with both complex order b and parameter β under subordination φ.

Motivated by these considerations, we introduce the class $H^{j}_{\delta,p,b}(\varphi)$. The results in Theorems 2.1–2.3 lead to the following observations. First, the bounds on the Fekete–Szegö functional show a non-linear dependence on the operator factor $G_n(\delta, p)$, order b , and parameter β . Second, the critical values ρ_1 , ρ_2 for real μ in Theorem 2.2 are directly modified by $D^{j}_{\delta,p}$, giving different transition points than the standard Sălăgean ($\delta = 1$) or univalent ($p = 1$) cases.

In this paper, we obtain the Fekete-Szegö inequality for the functions in the class $H^{j}_{\delta,p,b}(\varphi)$.

2. The Fekete-Szegö problem

Let Ω be the class of functions of the form :

$$w(z) = w_1 z + w_2 z^2 + w_3 z^3 + \cdots$$

In the open unit disk $\mathcal{U}$ satisfying $|w(z)| < 1$.

We need the following lemmas.

Lemma 2.1 [4] If $w \in \Omega$, then for any real number t

$$|w_2 - t w_1^2| \leq \begin{cases} -t & , \quad t \leq -1 \\ 1 & , -1 \leq t \leq 1 \\ t & , \quad t \geq 1 \end{cases}$$

The bounds are sharp. When $t < -1$ or $t > 1$, equality holds if and only if $w(z) = z$ or one of its rotations. If $-1 < t < 1$, then equality holds if and only if $w(z) = z^2$ or one of its rotations.

Equality holds for $t = -1$ if and only if

$$w(z) = z\frac{\alpha + z}{1 + \alpha z}, \qquad (0 \leq \alpha \leq 1)$$

or one of its rotations, while for $t = 1$, the equality holds if and only if

$$w(z) = -z\frac{\alpha + z}{1 + \alpha z}, \qquad (0 \leq \alpha \leq 1)$$

or one of its rotations.

Lemma 2.2 [1] If $w \in \Omega$,then for any complex number t

$$|w_2 - t w_1| \leq \max\{1, |t|\}$$

The result is sharp for function given by $w(z) = z$ or $w(z) = z^2$.

Now, we will prove coefficients bounds of the class $H^j_{\delta,p,1}(\varphi)$, where ($b = 1$) by making use of lemmas 2.1 and 2.2, as following.

2.1. Coefficient bounds for the subclass $H^j_{\delta,p,b}(\varphi)$

We first consider the case $b = 1$.

Theorem 2.1. Let $\varphi(z) = 1 + B_1 z + B_2 z^2 + \cdots$, where $B_n's$ are real with $B_1 > 0$, $B_2 \geq 0$, if $f \in \mathcal{A}(p)$ given by (1.4) belongs to $H^j_{\delta,p,1}(\varphi)$ and μ is a complex number, then:

$$\left|a_{p+2} - \mu a_{p+1}^2\right| \leq \frac{B_1\ p(1 + p\beta - \beta)}{2(1 + p\beta + \beta)G_2(\delta, p)} \max\{1, |v|\} \tag{2.1}$$

Where:

$$v = -\frac{B_2}{B_1} - pB_1 + \frac{2\mu\, p\, B_1 G_2(\delta,p)}{\left(G_1(\delta,p)\right)^2}\left[1 - \frac{\beta^2}{(1+\beta p)^2}\right]$$

and $G_1(\delta, p)$, $G_2(\delta, p)$ defined as (1.7).

Proof. If $f(z) \in H^j_{\delta,p,1}(\varphi)$, from definition 1.7 and eq. (1.16), there exists a Schwarz function

$$w(z) = w_1 z + w_2 z^2 + w_3 z^3 + \cdots \in \Omega$$

Such that

$$1 + \frac{1}{b}\left\{\frac{1}{p}\frac{z\left(D^j_{\delta,p}f(z)\right)' + \beta z^2\left(D^j_{\delta,p}f(z)\right)''}{(1-\beta)\left(D^j_{\delta,p}f(z)\right) + \beta z\left(D^j_{\delta,p}f(z)\right)'} - 1\right\} = \varphi\big(w(z)\big)$$

From (1.6) we have

$$\left(D^j_{\delta,p}f(z)\right)' = pz^{p-1} + \sum_{n=1}^{\infty}(p+n)G_n(\delta,p)\, a_{p+n}\, z^{p+n-1}$$

$$\left(D^j_{\delta,p}f(z)\right)'' = p(p-1)z^{p-2} + \sum_{n=1}^{\infty}(p+n)(p+n-1)G_n(\delta,p)\, a_{p+n}\, z^{p+n-2}$$

Using polynomial long division, we obtain :

$$1 + \frac{1}{b}\left\{\frac{1}{p}\frac{z\left(D^j_{\delta,p}f(z)\right)' + \beta z^2\left(D^j_{\delta,p}f(z)\right)''}{(1-\beta)\left(D^j_{\delta,p}f(z)\right) + \beta z\left(D^j_{\delta,p}f(z)\right)'} - 1\right\}$$
$$= 1 + \left[\frac{(1+p\beta)G_1(\delta,p)}{p(1+p\beta-\beta)}a_{p+1}\right]z + \left[\frac{2G_2(\delta,p)(1+p\beta+\beta)}{p(1+p\beta-\beta)}a_{p+2} - \frac{(1+p\beta)^2\left(G_1(\delta,p)\right)^2}{p(1+p\beta-\beta)^2}a_{p+1}^2\right]z^2 + \cdots \tag{2.3}$$

Where

$$\varphi\big(w(z)\big) = 1 + B_1 w_1 z + (B_2 w_1^2 + B_1 w_2)z^2 + \cdots \tag{2.4}$$

From (2.3) and (2.4):

$$B_1 w_1 = \frac{(1+p\beta)G_1(\delta,p)}{p(1+p\beta-\beta)}a_{p+1} \tag{2.5}$$

and

$$B_2 w_1^2 + B_1 w_2 = \frac{2G_2(\delta,p)(1+p\beta+\beta)}{p(1+p\beta-\beta)}a_{p+2} - \frac{(1+p\beta)^2\left(G_1(\delta,p)\right)^2}{p(1+p\beta-\beta)^2}a_{p+1}^2 \tag{2.6}$$

Therefore from (2.5) and (2.6)

$$a_{p+1} = \frac{pB_1 w_1(1+p\beta-\beta)}{(1+p\beta)G_1(\delta,p)}$$

$$a_{p+2} = \frac{p(1+p\beta-\beta)}{2G_2(\delta,p)(1+p\beta+\beta)}\left[\, B_1 w_2 + w_1^2(B_2 + pB_1^2)\,\right]$$

Hence, for μ a complex number

$$a_{p+2} - \mu a_{p+1}^2 = \frac{pB_1}{2G_2(\delta,p)}\frac{(1+p\beta-\beta)}{(1+p\beta+\beta)}\left[w_2 - v w_1^2\right] \tag{2.7}$$

Where:

$$v = -\frac{B_2}{B_1} - pB_1 + \frac{2\mu\, p\, B_1 G_2(\delta,p)}{\left(G_1(\delta,p)\right)^2}\left[1 - \frac{\beta^2}{(1+\beta p)^2}\right] \tag{2.8}$$

From lemma 2.2

$$\left|a_{p+2} - \mu a_{p+1}^2\right| \leq \frac{B_1\ p(1+p\beta-\beta)}{2G_2(\delta,p)(1+p\beta+\beta)}\max\{1,|v|\}. \qquad \blacksquare$$

Theorem 2.2. Under the same hypotheses on $\varphi(z)$, let μ be real. Define

$$\rho_1 = \frac{G_1^2(\delta,p)(1+\beta p)^2[(B_2 - B_1) + pB_1^2]}{2\,p\,B_1^2[\,(1+\beta p)^2 - \beta^2]G_2(\delta,p)}$$

$$\rho_2 = \frac{G_1^2(\delta,p)(1+\beta p)^2[(B_2 + B_1) + pB_1^2]}{2\,p\,B_1^2[\,(1+\beta p)^2 - \beta^2]G_2(\delta,p)}$$

if f given by (1.4) belongs to the class $H^{j}_{\delta,p,1}(\varphi)$, where ($b = 1$) and $G_1(\delta,p)$, $G_2(\delta,p)$ defined as (1.7), then

$$|a_{p+2} - \mu a_{p+1}^2| \leq \begin{cases} \psi_1 & ; \quad \mu \leq \rho_1 \\ \psi_2 & ; \quad \rho_1 \leq \mu \leq \rho_2 \\ \psi_3 & ; \quad \mu \geq \rho_2 \end{cases}$$

Where :

$$\psi_1 = \frac{p}{2\,G_2(\delta,p)}\frac{(1+\beta p-\beta)}{(1+\beta p+\beta)}\cdot \left[pB_1^2 + B_2 - \frac{2\mu\,B_1^2\,p\,G_2(\delta,p)}{\left(G_1(\delta,p)\right)^2}\left(1-\frac{\beta^2}{(1+\beta p)^2}\right)\right]$$

$$\psi_2 = \frac{pB_1}{2\,G_2(\delta,p)}\frac{(1+\beta p-\beta)}{(1+\beta p+\beta)}$$

$$\psi_3 = \frac{p}{2\,G_2(\delta,p)}\frac{(1+\beta p-\beta)}{(1+\beta p+\beta)}\cdot \left[-pB_1^2 - B_2 + \frac{2\mu\,B_1^2\,p\,G_2(\delta,p)}{\left(G_1(\delta,p)\right)^2}\left(1-\frac{\beta^2}{(1+\beta p)^2}\right)\right]$$

Proof. from (2.7) and applying lemma 2.1

$$|a_{p+2} - \mu a_{p+1}^2| \leq \frac{pB_1}{2\,G_2(\delta,p)}\frac{(1+\beta p-\beta)}{(1+\beta p+\beta)}\begin{cases} -v & ; \quad v \leq -1 \\ 1 & ; \quad -1 \leq v \leq 1 \\ v & ; \quad v \geq 1 \end{cases}$$

Where v defined as (2.8) .

Hence, by rearranging the terms, the proof is complete.

■

Now, we will apply lemma 2.2 on the class $H^{j}_{\delta,p,b}(\varphi)$, we get the following theorem.

Theorem 2.3 Let $\varphi(z) = 1 + B_1 z + B_2 z^2 + \cdots$, where $B_n's$ are real with $B_1 > 0$, $B_2 \geq 0$, if $f(z)$ given by (1.4) belongs to the class $H^{j}_{\delta,p,b}(\varphi)$ and μ is a complex number, then:

$$|a_{p+2} - \mu a_{p+1}^2| \leq \frac{B_1\,|b|\,p(1+p\beta-\beta)}{2(1+p\beta+\beta)G_2(\delta,p)}\max\{1,|v|\}$$

Where:

$$v = -\frac{B_2}{B_1} - |b|\,pB_1 + \frac{2|b|\,pB_1\mu\,G_2(\delta,p)}{G_1^2(\delta,p)}\left(1-\frac{\beta^2}{(1+p\beta)^2}\right) \tag{2.9}$$

And $G_1(\delta,p)$, $G_2(\delta,p)$ defined as (1.7).

Proof. If $f(z) \in H^{j}_{\delta,p,b}(\varphi)$, then by Definition 1.7 and equation (1.16), there exists a Schwarz function

$$w(z) = w_1 z + w_2 z^2 + w_3 z^3 + \cdots \in \Omega$$

such that:

$$1 + \frac{1}{b}\left(\frac{1}{p}\frac{z\left(D^{j}_{\delta,p}f(z)\right)' + \beta z^2\left(D^{j}_{\delta,p}f(z)\right)''}{(1-\beta)D^{j}_{\delta,p}f(z) + \beta z\left(D^{j}_{\delta,p}f(z)\right)'} - 1\right) = \varphi\big(w(z)\big)$$

From the definition of the generalized Al-Oboudi operator in (1.6), we have:

$$D^{j}_{\delta,p}f(z) = z^p + G_1(\delta,p)a_{p+1}z^{p+1} + G_2(\delta,p)a_{p+2}z^{p+2} + \cdots$$

Differentiating with respect to z yields:

$$\left(D^{j}_{\delta,p}f(z)\right)' = pz^{p-1} + (p+1)G_1(\delta,p)a_{p+1}z^p + (p+2)G_2(\delta,p)a_{p+2}z^{p+1} + \cdots$$

$$\left(D^{j}_{\delta,p}f(z)\right)'' = p(p-1)z^{p-2} + (p+1)pG_1(\delta,p)a_{p+1}z^{p-1} + (p+2)(p+1)G_2(\delta,p)a_{p+2}z^{p} + \cdots$$

Using polynomial series division on the left-hand side, we get:

$$1 + \frac{1}{b}\left(\frac{1+p\beta}{p(1+p\beta-\beta)}G_1(\delta,p)a_{p+1}z + \left[\frac{2(1+p\beta+\beta)}{p(1+p\beta-\beta)}G_2(\delta,p)a_{p+2} - \frac{(1+p\beta)^2}{p(1+p\beta-\beta)^2}G_1^2(\delta,p)a_{p+1}^2\right]z^2 + \cdots\right)$$

Expanding the right-hand side $\varphi\big(w(z)\big)$gives:

$$\varphi\big(w(z)\big) = 1 + B_1 w_1 z + (B_1 w_2 + B_2 w_1^2)z^2 + \cdots$$

Equating coefficients of corresponding powers of $z$and z^2on both sides:

$$\frac{1}{b}\frac{(1+p\beta)}{p(1+p\beta-\beta)}G_1(\delta,p)a_{p+1} = B_1 w_1$$

Solving for a_{p+1}:

$$a_{p+1} = \frac{b\,pB_1(1+p\beta-\beta)}{(1+p\beta)G_1(\delta,p)}w_1 \tag{2.10}$$

And

$$\frac{1}{b}\left[\frac{2(1+p\beta+\beta)}{p(1+p\beta-\beta)}G_2(\delta,p)a_{p+2} - \frac{(1+p\beta)^2}{p(1+p\beta-\beta)^2}G_1^2(\delta,p)a_{p+1}^2\right] = B_1 w_2 + B_2 w_1^2$$

Substituting $a_{p+1}^2$from (2.10) and solving for a_{p+2}:

$$a_{p+2} = \frac{b\,p(1+p\beta-\beta)}{2(1+p\beta+\beta)G_2(\delta,p)}\left[B_1 w_2 + B_2 w_1^2 + \frac{b\,pB_1^2}{(1+p\beta-\beta)^2}w_1^2\right]$$

Now, computing the Fekete–Szegö functional:

$$a_{p+2} - \mu a_{p+1}^2 = \frac{b\,pB_1(1+p\beta-\beta)^2}{2(1+p\beta+\beta)G_2(\delta,p)}(w_2 - v w_1^2)$$

where v is given by (2.9) .

Taking the absolute value on both sides and applying Lemma 2.2 :

$$|a_{p+2} - \mu a_{p+1}^2| \leq \frac{|b|\,pB_1(1+p\beta-\beta)^2}{2(1+p\beta+\beta)G_2(\delta,p)}\max\{1,|v|\}$$

This completes the proof of Theorem 2.3. ■

Conclusion

In this paper, we have successfully introduced and investigated a novel subclass of analytic multivalent functions, denoted by $H_{\delta}^{j}{}_{,p,\mathrm{b}}(\varphi)$, utilizing the generalized Al-Oboudi differential operator in the open unit disk $\mathcal{U}$. By applying the classical properties of the Schwarz function and subordination principles, we established sharp upper bounds for the Fekete-Szegő for functions belonging to this comprehensive class. Furthermore, we demonstrated that our main theorems directly generalize and unify several prominent results previously obtained in geometric function theory. The structural framework presented herein opens new avenues for further investigation into other coefficient problems, such as Hankel determinants and growth estimates for related operators.

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