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Nonlinear Partial Differential Equations in Fractal Spaces

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Index Terms
Nonlinear PDEs, Fractal spaces, Soliton solutions, Local fractional derivatives, Optical solitons, M-truncated derivative.

Abstract
Nonlinear partial differential equations (NLPDEs) in fractal spaces create a strong mathematical structure which enables scientists to study wave propagation and soliton movements and strange particle movement across self-similar irregular shapes. The current paper presents a comprehensive study which includes multiple exact soliton solutions which researchers obtained through local fractional derivatives and M-truncated derivatives and advanced analytical methods which include the extended (G′/G)-expansion method and ϕ^6-model expansion and Lie symmetry reduction and auxiliary equation approaches. We transform classical models which include the Korteweg-de Vries (KdV) and nonlinear Schrödinger (NLS) and Boussinesq and Kodama and cubic-quintic non-paraxial pulse equations into fractal spaces to obtain bright and dark and singular and periodic and combined and optical soliton molecules. The research paper provides a comprehensive examination of how stability testing and fractal order changes influence wave patterns. The results of this study support existing findings from 2025 which focus on nonlinear dispersive and optical systems while providing better understanding of surface geometry and shallow-water acoustics and optical fiber technologies in fractal materials.

المعادلات التفاضلية الجزئية غير الخطية في الفضاءات الكسرية

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الكلمات المفتاحية
المعادلات التفاضلية الجزئية غير الخطية، الفضاءات الكسرية، حلول السوليتونية، المشتقات الكسرية المحلية، السوليتونات البصرية، مشتقة M-المبتورة.

الملخص
تشكّل المعادلات التفاضلية الجزئية غير الخطية (**NLPDEs**) في الفضاءات الكسرية بنية رياضية قوية تمكّن العلماء من دراسة انتشار الموجات، وديناميكا السوليتونات (**Solitons**)، وحركة الجسيمات الغريبة عبر الهياكل غير المنتظمة ذاتية التشابه. تقدم الورقة الحالية دراسة شاملة تتضمن حلولاً سوليتونية دقيقة متعددة تم الحصول عليها عبرالمشتقات الكسرية المحلية، ومشتقات **M**- المبتورة (**M-truncated derivatives**)، والأساليب التحليلية المتقدمة التي تشمل طريقة التوسع (**G'/G**) الممتدة، وتوسيع نموذج ϕ^6، واختزال تناظر لي (**Lie symmetry**)، ونهج المعادلات المساعدة. لقد تم تحويل النماذج الكلاسيكية التي تشمل معادلات كورتيويج- دي فريس (**KdV**)، وشرودنجر غير الخطية (**NLS**)، وبوسينيسك (**Boussinesq**)، وكوداما (**Kodama**)، ومعادلات النبض غير المحورية التكعيبية-الخماسية (**cubic-quintic**) إلى فضاءات كسرية للحصول على جزيئات سوليتونية مضيئة، ومظلمة، ومفردة، ودورية، ومدمجة، وبصرية.
علاوة على ذلك، وتوفر الورقة البحثية فحصاً شاملاً لكيفية تأثير اختبار الاستقرار وتغيرات الرتبة الكسرية على الأنماط الموجية. تدعم نتائج هذه الدراسة النتائج الحالية لعام 2025 والتي تركز على الأنظمة البصرية والتشتتية غير الخطية، مع توفير فهم أفضل لهندسة الأسطح، وصوتيات المياه الضحلة، وتقنيات الألياف البصرية في المواد الكسرية.

1. INTRODUCTION

Fractal spaces are one of the primary jobs of classical Euclidean geometry, in which the dimensionality is not restricted to an integer scale but rather represented by non-integer Hausdorff dimensions. This geometric complexity brings about intrinsic irregularity and scale invariance which is a characteristic of most systems in nature and engineered systems. Physical processes in these environments are anomalously scaled, long-range dependent and experience memory effects which cannot be well modeled with standard integer-order differential operators. The mathematical modeling of nonlinear processes in fractal media therefore requires the construction and use of the generalized calculus frameworks, especially the local fractional derivatives

and M-truncated operators, which give the required apparatus to tackle the complexities in a serious way.

The weakness of classical partial differential equations (PDEs) can be seen in cases where one is trying to model transport, diffusion, and propagation of waves in a fractal or heterogeneous media. In the standard models, there is a smoothness and locality assumption but in fractal systems, discontinuities and nonlocal interactions always occur. Fractional calculus is a mathematical extension of differentiation to non-integer orders, so that it is possible to include memory kernels and hereditary properties in governing equations. This extension has been found to be critical in modeling the systems whose modeling has been difficult to do through the traditional approach including porous media flow, viscoelastic materials and in the models of anomalous diffusion. More recent extensions also add operators like the M-truncated derivative, which helps to better represent fractal dynamics, and to better resolve the analytical tractability of nonlinear systems, as is used in optical models and fractal wave models (Li, 2025; Kaya Sağlam, 2025).

Nonlinear partial differential equations (NLPDEs) take center stage in the study of nonlinear science since they can be used in describing how waves interact, the formation of patterns and the transfer of energy in complex systems. On fractal schemes, these equations have more elaborate dynamical structures than their classical analog. The addition of fractional and fractal operators causes the appearance of various families of solutions such as bright solitons, dark solitons, periodic waves and quasi periodical structures. These are not purely mathematical products, but are related to physically measurable effects in optical fibers, shallow water systems and acoustic media. As an example, modification of soliton amplitude, velocity and stability properties are found when fractional derivatives are incorporated into nonlinear Schrodinger-type models (Abdalla et al., 2025; Jlali et al., 2025).

Recent developments of the techniques of analysis and semi-analysis have made it easy to construct the precise solutions to complex NLPDEs in fractal environments. The extended method is one of such. (G'/G) The approaches to trial function, Lie symmetry analysis, and variational iteration methods have been found to be useful in deriving explicit wave structures. These methodologies do not only give closed-form solutions but also give an insight about the qualitative behavior of non linear systems. As an example, the Lie symmetry analysis has been effectively used to analyze perturbed shear flow models with finding invariant solutions and symmetry reductions simplifying high-dimensional problems (Mandal et al., 2025). Equally, long-term analytical techniques have been employed to understand soliton dynamics and chaotic states in nonlinear field equations like the LandauGinzburgHiggs model with the emphasis on nonlinearity and dispersion (Mehdi et al., 2025).

The applicability of fractal and fractional NLPDEs is applicable in various fields of science. In surface geometry, the coupled nonlinear systems explain the effect of curvature dynamics and interface dynamics, where fractal properties affect the propagation of waves and their stability (Farooq et al., 2025a). Under different environmental conditions, integrable equations like the SchwarzKortewegdeVries and the Boussinesq models can be used to model the behaviour of dispersive waves and energy transport in shallow water and acoustic media (Farooq et al., 2025b; Farooq et al., 2025c). Another important area of application is in optical physics, in which nonlinear pulse propagation models with cubic and quintic nonlinearities and fractional operators can give accurate descriptions of the dynamics of ultrashort pulses in fibers and waveguides (Yokus and Isah, 2025; Rizvi et al., 2025a). These models are of special significance in the dispersion management, soliton interaction, and creation of stable optical structures.

Non-integrable models are also very important besides the integrable systems in describing realistic physical situations in which not all models are guaranteed to be solvable. Complex behaviors such as chaos, bifurcations as well as multi-soliton interactions are frequently found in the study of such systems. Numerical studies tend to be analytical and thus facilitate stability regime descriptions and sensitivity to parameters. As an illustration, the Yu-Toda-Sasa-Fukuyama equation has been studied to find stability condition and solitary wave solutions to nonlinear and dispersive effects (Rizvi et al., 2025b). Likewise, time-dependent perturbations of equations of the Korteweg–de Vries type also create the varieties of fluctuation which can greatly change the wave profiles and properties of propagation (Alquran, 2025).

The other valuable aspect of the fractional modeling is that it can be applied to the biological and epidemiological systems where memory effects and nonlocal interactions are related. It has been shown that the dynamics of the viral transmission can be modeled by the use of fractional variational iteration methods, which show the flexibility of the concept of the fractional calculus in the context of the classical physical systems (Manivel et al., 2025). These applications support the relevance of the idea of fractional operators, although they are not related to the problem of wave propagation, to taking into account the complex time behavior and long-term dependencies.

Optical soliton has found a lot of interest in the study of optical solitons in the framework of fractional and fractal because of its implications on technology. The optical solitons are used as information carriers in communication systems which are stable in the propagation of information and their actions under the fractional dynamics gives new possibilities to control the propagation of pulses. The studies of cubic–quintic nonlinear models have found that soliton molecules, chirped pulses, and quasi-periodic solutions exist, and they depend on the order of the fractional derivative and the underlying fractal structure (Rizvi et al., 2025c; Akbar et al., 2023). Other previous works on non-paraxial pulse propagation and Airy pulse evolution also emphasize the variety of optical processes that can be simulated in this system (Khater, 2022a; Zhong et al., 2015).

Even with such developments, systematic and comprehensive investigation of precise solutions of fractal NLPDEs is still needed. Numerous studies that exist are dealing with only a single equation or restrictive sets of solution, which has created a gap in the general knowledge of the structure of solutions in various models. Moreover, the interplay between nonlinear dynamics and fractal geometry is yet to be completely described, especially in the aspects of the effect of fractal dimensions in the stability of solutions and the morphology of waves. To overcome these gaps, it is important that unified analytical frameworks should be developed to produce a wide range of solutions, which are both mathematically rigorous and physically relevant.

The study seeks to improve the developing field through its research which develops deeper mathematical knowledge and broader mathematical understanding of exact solution methods for essential nonlinear partial differential equations that occur in fractal environments. The advanced analytical techniques together with generalized fractional operators create multiple solution types which include soliton solutions and periodic solutions and quasi-periodic solutions. The models considered in this work span multiple application domains, including surface geometry, shallow water waves, acoustic media, and optical pulse propagation, ensuring both theoretical significance and practical applicability.

The work presents its original contribution through an integrated framework which links fractal calculus with nonlinear analysis and exact solution methods to create a complete understanding of wave behavior in complex media. The study presents a systematic examination of both integrable and non-integrable models to demonstrate how fractional parameters and perturbations and nonlinear coefficients affect solution outcomes. The method extends current findings and it reveals entirely new groups of solutions together with movement patterns which researchers have never documented before in existing research.

The introduction of fractal and fractional operators into nonlinear partial differential equations creates a major breakthrough for complex system modeling. The formulations describe essential system characteristics through memory effects and nonlocal interactions and anomalous scaling which classical models fail to capture. The study shows exact solutions and their properties which results in a better understanding of how nonlinear waves behave in fractal environments while establishing the base for upcoming studies in this fast-evolving field.

1.1 Research Gap and Critical Analysis

The recent research on nonlinear partial differential equations which study fractal spaces has primarily studied two mathematical areas which involve single mathematical models or specific types of exact solutions. The research studies investigated local fractional derivatives together with M-truncated operators but only two studies managed to create a unified system that combined both methodologies (Li, 2025; Kaya Sağlam, 2025).

The previous research studies focused on developing analytical wave structures which lacked adequate explanations about their application to physical systems and their stability patterns and their nonlinear interactions with basic fractal components in heterogeneous environments (Mandal et al., 2025; Mehdi et al., 2025).

The existing literature presents another limitation because researchers have not compared integrable systems with perturbed nonlinear systems under fractional system conditions. The researchers studied single-equation models but they did not investigate how different fractional values affect wave propagation and soliton stability in various physical situations (Farooq et al., 2025b; Rizvi et al., 2025b).

The current study uses a unified analytical framework to resolve its research questions by linking multiple nonlinear fractal equations through their local fractional and M-truncated operator components. The current study creates direct analytical relationships between precise wave patterns and their real-world usage in optical fibers and shallow-water systems and acoustic signal propagation and network transmission systems (Akbar et al., 2023; Rizvi et al., 2025c).

2. FUNDAMENTAL DEFINITIONS IN FRACTAL CALCULUS

Fractal calculus establishes a precise mathematical system which allows researchers to study functions that exist on irregular domains which have non-integer dimensional measurements. The mathematical system proves effective for modeling systems which demonstrate unusual size patterns and contain abrupt changes and extended dependency relationships. Fractal calculus introduces new mathematical operators which enable the measurement of the fundamental complexity present in fractal materials. The local fractional derivative serves as a fundamental operator which researchers use to analyze functions that maintain continuous movement yet lack conventional differentiation capabilities (Li, 2025; Kaya Sağlam, 2025).

$$D_x^{\alpha}u(x) = \lim_{\Delta x \to 0} \frac{u(x+\Delta x) - u(x)}{(\Delta x)^{\alpha}\Gamma(1+\alpha)}$$

Thelocal fractional derivative of order $\alpha\alpha$ which exists between $0 < \alpha \leq 1$, provides a generalized rate of change which depends on fractional increments because the derivative definition maintains its validity for functions that exhibit extreme irregularity. The inclusion of the Gamma function $\Gamma(1+\alpha)$ guarantees proper normalization and dimensional consistency, while the fractional power of the increment reflects the scaling properties of fractal structures. The operator demonstrates strong capabilities to accurately simulate physical processes which include anomalous diffusion and wave propagation in heterogeneous media and transport phenomena in porous structures because classical derivatives do not deliver precise results (Abdalla et al., 2025; Rizvi et al., 2025a).

The local fractional derivative operates according to mathematical principles which maintain core mathematical functions like linearity and product rules, but the application of these principles leads to results which differ from standard rules because of the operator's nonlocal characteristics. The operator uses fractional increments to establish micro-scale sensitivity which enables it to detect tiny structural changes that exist within fractal domains. Nonlinear partial differential equations which use fractal spaces for their definition now depend on this technique as their core method (Farooq et al., 2025a; Farooq et al., 2025b).

Dynamic models need temporal generalizations which enable them to handle memory effects and hereditary characteristics, in addition to spatial fractional operators. The M-truncated derivative creates an advanced method which combines Mittag-Leffler function with differentiation to achieve accurate time-dependent fractional behavior results.

$$D_{M,t}^{\alpha,\beta}u(t) = lim_{\tau\to 0}\frac{u\left(tE_{\beta}(\tau t^{1-\alpha})\right) - u(t)}{\tau}$$

The M-truncated derivative extends its flexibility to additional parameters α and β, which work together with the truncated Mittag-Leffler function that serves as a generalized exponential kernel to manage memory loss. The analysis shows that this method improves how researchers study nonlinear optical models and fractional Schrödinger equations (Kaya Sağlam, 2025; Li, 2025).

The operator demonstrates its ability to effectively model nonlinear systems which track present state through past state dependencies, which includes systems like optical pulse propagation and viscoelastic dynamics. The operator serves as a critical component which helps achieve precise solutions for fractional nonlinear equations (Akbar et al., 2023; Rizvi et al., 2025c).

The local fractional derivatives with M-truncated operators create a unified mathematical system which can model both spatial irregularity and temporal memory. This system enables scientists to study complex nonlinear behaviors that emerge in environments with fractal patterns.

3. Nonlinear Models in Fractal Spaces and Their Solutions

Nonlinear partial differential equations in fractal spaces represent an advanced extension of classical wave equations which use fractional operators to model both anomalous dispersion and nonlocal interactions together with scale-dependent behavior, resulting in more complex dynamic behavior than traditional models (Mehdi et al., 2025; Mandal et al., 2025).

The introduction of fractional derivatives alters both the nonlinear and dispersive components, which results in major alterations to the wave propagation characteristics that affect velocity and stability and waveform structure. The researchers used analytical methods which include traveling wave transformations and symmetry analysis to transform these equations into simpler forms that enable them to discover exact solutions (Alquran, 2025; Rizvi et al., 2025b).

Researchers use these models to investigate surface geometry and shallow water dynamics and acoustic media and nonlinear optics because fractional operators enable more precise modeling of actual physical systems (Farooq et al., 2025a; Yokus and Isah, 2025).

TABLE I. Comparison of Fractal Nonlinear Models and Their Applications

Model	Fractional Operator	Solution Type	Physical Application
Fractional KdV Equation	Local Fractional Derivative	Bright and Dark Solitons	Shallow Water Waves
Fractional Schrödinger Equation	Cubic-Quintic Fractional Operator	Optical Solitons	Optical Fibers
Fractal Boussinesq Equation	Local Fractional Derivative	Bell-shaped Solitons	Acoustic Systems
Kodama Equation	M-Truncated Derivative	Chirped Pulses	Signal Transmission

The table summarizes the main nonlinear fractal models investigated in this study together with their corresponding fractional operators, analytical wave structures, and practical physical applications.

3.1 Local Fractional KdV Equation

The local fractional Korteweg–de Vries equation presents an extension of the classical KdV model through its application in fractal spaces and its use of fractional derivatives to model dispersive wave propagation in complex materials.

$$D_t^{\alpha u} + 6u\,D_x^{\alpha u} + D_x^{3\alpha u} = 0$$

The fractional parameter α in this formulation controls system scaling properties and nonlocal behavior whereas the nonlinear term models wave interactions and the dispersive term govern wave propagation. This equation has been widely used to model shallow water waves and acoustic phenomena under fractional dynamics (Farooq et al., 2025b; Farooq et al., 2025c).

The application of a traveling wave transformation produces exact solutions because it transforms the equation into an ordinary differential equation which uses the similarity variable.

$$\xi = k\frac{x^{\alpha}}{\Gamma(1+\alpha)} - c\frac{t^{\alpha}}{\Gamma(1+\alpha)}$$

The transformation maintains the fractal properties of the variables which allows researchers to develop analytical soliton solutions through established techniques that include expansion methods and trial function approaches (Rizvi et al., 2022; Tariq et al., 2021).

The bright soliton solution shows a localized wave that carries its highest energy at the center point and diminishes exponentially to both sides which occurs in systems that exhibit nonlinear focusing behavior (Zhong et al., 2015; Khater, 2022a).

$$u(x,t) = 2c\,sech^2(2c\xi)$$

The dark soliton solution creates a specific wave pattern which produces a localized wave amplitude drop inside a continuous background. This effect usually occurs in defocusing media and nonlinear optical systems. (Yokus and Isah, 2025; Abdalla et al., 2025).

$$u(x,t) = \frac{c}{2}\left(1 - tanh^2\left(\frac{\sqrt{c}}{2}\xi\right)\right)$$

The singular soliton solution now exists as a mathematical entity which exhibits divergent behavior at certain points because of the hyperbolic cosecant function, which makes this solution appropriate for modeling extreme localized events (Mehdi et al., 2025).

$$u(x,t) = \frac{c}{2}csch^2\left(\frac{\sqrt{c}}{2}\xi\right)$$

The combined bright-dark soliton solution represents a hybrid structure which combines localized peaks and dips to show multiple nonlinear effects that exist together in the system (Rizvi et al., 2025c; Akbar et al., 2023).

$$u(x,t) = A[sech^2(\mu\xi) - Btanh^2(\mu\xi)]$$

The analytical solutions show that the local fractional KdV equation can accurately model different nonlinear wave patterns which occur in fractal environments because the fractional parameter enables additional control of wave properties including amplitude and velocity and stability, making the equation crucial for fluid dynamics and optics and acoustic system applications.

3.2 Fractal Nonlinear Schrödinger Equation with Cubic-Quintic Nonlinearity

The nonlinear Schrödinger equation serves as a basic framework which describes how waves travel through nonlinear dispersive materials that exist in optical fibers and waveguides, while its extension to fractal spaces through fractional operators enables the modeling of both anomalous dispersion and nonlocal interactions and memory effects, which scientists need to study ultrashort optical pulse behavior (Yokus and Isah, 2025; Rizvi et al., 2025c).

$$i\,D_t^{\alpha}\psi + D_x^{2\alpha}\psi + (|\psi|^2 + \gamma|\psi|^4)\psi + \delta D_x^{2\alpha}(|\psi|^2)\psi = 0$$

The presence of cubic and quintic nonlinear terms demonstrates that realistic optical media show their highest-order nonlinear effects, which occur during situations of extreme intensity. The cubic term represents Kerr-type nonlinearity while the quintic term establishes limits, which prevent the wave amplitude from increasing without restraint. The additional non-paraxial correction term involving $\delta D_x^{2\alpha}(|\psi|^2)\psi$ enables the model to include advanced dispersion effects together with nonlocal spatial effects, which enable precise modeling of optical pulse movement beyond the basic paraxial method (Khater, 2022a; Akbar et al., 2023).

To achieve analytical solutions for the problem at hand, researchers use a traveling wave ansatz, which changes the governing equation into an ordinary differential equation that describes a similarity variable, because this transformation maintains fractal scaling properties for both space and time measurements, it allows researchers to discover direct wave solutions.

$$\psi(x,t) = \phi(\xi)e^{i(\kappa x^{\alpha} - \omega t^{\alpha})}$$

This ansatz separates the wave's amplitude and phase components which enables researchers to study envelope dynamics and phase evolution as two separate elements. The reduced equation can be solved through analytical methods which include expansion methods and trial function approaches that have previously demonstrated success in solving fractional optical models (Rizvi et al., 2022; Tariq et al., 2021).

The bright optical soliton solution describes a localized pulse which exhibits peak amplitude that reduces through an exponential function from its central point, this solution shows a relationship with focusing nonlinearities which enable stable energy localization throughout optical fibers (Zhong et al., 2015; Abdalla et al., 2025).

$$\psi(x,t) = \sqrt{\frac{2\beta}{\gamma}}sech(\mu\xi)exp(i(\kappa x^{\alpha} - \omega t^{\alpha}))$$

The dark optical soliton solution creates a dark wave solution which creates a concentrated wave pattern that remains fixed within its surrounding continuous background, this type of solution is characteristic of defocusing nonlinear media and is widely observed in nonlinear optical systems with normal dispersion (Yokus and Isah, 2025; Akbar et al., 2023).

$$\psi(x,t) = \sqrt{\frac{-\beta}{\gamma}}tanh(\mu\xi)exp(i(\kappa x^{\alpha} - \omega t^{\alpha}))$$

The optical soliton molecule displays a more complex structure which contains multiple soliton components that interact with each other to create a stable bound state. The structure exists because nonlinear interactions and dispersion forces reach an equilibrium state. The system serves an essential function in advanced optical communication systems which depend on multi-pulse stability (Rizvi et al., 2025c; Mehdi et al., 2025).

$$\psi(x,t) = [A1\,sech(\mu 1\xi) + A2\,sech(\mu 2\xi)cos(\nu\xi)]exp(i\theta)$$

Multiple parameters include $A1, A2, \mu 1, \mu 2, \nu$ which enable precise management of soliton component interactions between two different solitons while creating complex wave patterns which display periodic modulation and quasi-bound states. The combination of fractal operators with higher-order nonlinearities creates dynamic results which these solutions show.

The fractal nonlinear Schrödinger equation with cubic-quintic nonlinearity functions as an all-encompassing model which researchers use to investigate nonlinear wave behavior in intricate materials because the model's fractional derivatives and non-paraxial elements enable it to better approximate actual physical behavior especially in optical systems which display dispersion and nonlinearity and memory effects that interact through extremely nonlinear processes.

3.3 Fractal Boussinesq Equation

The Boussinesq equation serves as a primary mathematical framework which describes nonlinear wave movement between two directions on shallow water surfaces and elastic materials, because its solution extends to fractal space through local fractional derivatives which enable the modeling of nonstandard dispersion and spatial dispersion and different environmental characteristics that operate at varying distances. The mathematical description of wave dynamics in fractal media requires fractional operators because classical assumptions about smoothness and uniformity cannot be applied to these environments (Farooq et al., 2025c; Farooq et al., 2025d).

$$D_t^{2\alpha u} - D_x^{2\alpha u} - D_x^{2\alpha}(u^2) - D_x^{4\alpha u} = 0$$

The equation uses fractional time derivative $D_t^{2\alpha u}$ to describe how the wave field evolves through time and how its spatial components function. The spatial fractional derivatives show how wave functions experience both nonlinear wave propagation and advanced wave dispersal. The nonlinear term $D_x^{2\alpha}(u^2)$ describes how waves interact through their different amplitude levels, while the fourth-order fractional derivative $D_x^{4\alpha u}$ determines how the wave profile spreads and maintains its stability. The combination of these elements establishes how localized stable structures develop while it controls the shift from linear to nonlinear behavior (Rizvi et al., 2025b; Mehdi et al., 2025).

The equation's most important aspect allows it to produce bell-shaped soliton solutions, which describe localized wave packets that retain their original form across lengthy distances through the equilibrium between nonlinearity and dispersion. These structures serve as crucial tools for simulating shallow water wave behavior and acoustic wave propagation and nonlinear elastic waves which occur in fractal environments.

$$u(x,t) = A\,sech^2(B(x^\alpha - ct^\alpha))$$

The parameters A and B control the soliton's amplitude and width whereas the fractional parameter α α enables additional solution scaling control. The wave profile deforms and its propagation speed changes as α moves away from unity because fractal geometry affects wave movement. Recent studies confirmed this behavior through research on nonlinear dispersive waves and fractional Boussinesq-type systems (Farooq et al. 2025c; Rizvi et al. 2025b).

3.4 Nonlinear Kodama Equation with M-Truncated Derivative

The Kodama equation serves as a common mathematical tool for studying nonlinear dispersive wave behavior in both fluid dynamics and optical systems. The M-truncated derivative generalization of the equation establishes advanced mathematical complexity through its combination of fractal scaling and temporal memory effects. The equation formulation enables the model to describe nonlocal spatial interactions together with time-dependent hereditary behavior, which scientists consider essential for their studies of complex nonlinear systems (Li, 2025; Kaya Sağlam, 2025).

$$i\,D_{M,t}^{\alpha,\beta}u + D_{M,x}^{2\alpha,\beta}u + |u|^2 u = 0$$

The system's time progression changes because the system uses Mittag-Leffler function as a memory decay exponent which acts as a general exponential decay function. The parameters α and β control how nonlocality behaves between both memory loss and memory storage within wave field amplitude and phase behavior.

The nonlinear term $|u|^2 u$ represents self-phase modulation, a key mechanism in nonlinear optics, while the fractional spatial derivative $D_{M,x}^{2\alpha,\beta}u$ governs dispersion under fractal conditions. The combination results in multiple optical wave patterns that include chirped pulses and quasi-periodic solutions and localized solitons. The transmission of signals through optical communication systems receives better control through the frequency range which chirped pulses create as their frequency changes throughout their traveling distance (Akbar et al., 2023; Rizvi et al., 2025c).

The fractional parameters which affect these solutions enable researchers to control wave properties. which makes M-truncated Kodama equation an effective method to investigate nonlinear wave behavior in complicated media. The analytical tractability of this formulation enables researchers to derive exact solutions which help explain how the system will operate (Li, 2025; Kaya Sağlam, 2025).

3.5 Perturbed KdV and Coupled Systems

The perturbed fractional Korteweg–de Vries equation extends the classical KdV model by introducing a time-dependent perturbation term, this modification allows the equation to account for external influences such as variable environmental conditions, forcing mechanisms, and temporal fluctuations in system parameters. Practical physical systems need these extensions because their behaviors do not conform with idealized assumptions (Alquran, 2025; Rizvi et al., 2025b).

$$D_t^{\alpha u} + 6u\,D_x^{\alpha u} + D_x^{3\alpha u} + \epsilon(t)u = 0$$

The perturbation function ϵ(t) creates two effects which create changes in wave amplitude and stability of the system. The system demonstrates two distinct dynamic patterns because it experiences three types of instability and two different soliton states which system undergoes. The term enhances solution space which

enables the model to represent more physical phenomena that show up in fluid dynamics and nonlinear optics (Alquran, 2025).

Coupled nonlinear systems exist in multiple physical situations which involve surface geometry and multi-component fluids and optical wave interactions. The multiple-field systems describe how different fields interact with each other while they permit vector soliton solutions which show coupled wave structures that have separate but connected parts (Farooq et al., 2025a; Mehdi et al., 2025).

$$u(x,t) = \begin{pmatrix} a1\backslash sech(\xi) \\ a2\ \tanh(\xi) \end{pmatrix}$$

The vector soliton solution shows that two different types of energy components exist together in a system that uses coupled components which develop their own energy patterns but stay interconnected through their interaction mechanisms. The solutions help to explain how optical fibers and coupled waveguides and fluid systems experience polarization effects.

Researchers who study perturbed and coupled fractional systems gain better understanding of how Nonlinear waves interact with each other, stability mechanism's function, and energy transfer processes occur while fractional calculus shows its value for modeling Complex real-world situations.

4. STABILITY ANALYSIS

The physical validity of nonlinear wave solutions in fractal spaces needs to prove their stability against disturbances because this property serves as a basic requirement for their physical existence. Fractal systems differ from classical integer-order systems because they generate disturbance propagation through nonlocal temporal behavior, which results in different evolutionary patterns. The modified stability criteria require explicit fractional order parameter α to be understood, which describes both memory effects and anomalous scaling (Abdalla et al., 2025; Rizvi et al., 2025c).

The process of analyzing linear stability begins with adding a small disturbance to the exact solution $u0$ the perturbed solution is defined through this relationship.

$$u = u0 + \epsilon v e^{\lambda t \alpha}$$

The equation shows ϵ as a small parameter and v as the perturbation mode while λ represents the growth rate. The exponential term $e^{\lambda t \alpha}$ describes the fractal temporal evolution through fractional time exponents which change standard exponential growth and decay patterns. The characteristic equation emerges from the governing fractional equation through ϵ linearization of the system which generates a fractal dispersion relation.



Figure 1. Effect of Fractional Order α on Soliton Stability

The graph shows how fractional order α values affect the stability of nonlinear soliton solutions. Lower fractional values improve wave localization and reduce perturbation growth rates, which results in enhanced stability for fractal nonlinear systems.

The fractal dispersion relation differs from the classical one because the system contains fractional derivatives which create nonlocal memory effects. These effects distribute energy across multiple modes which leads to increased stability across multiple parameter settings. The perturbation growth rate decreases between $0 < \alpha < 1$ while $\alpha = 1$ represents the standard case. The fractional optical models demonstrate this behavior because their soliton solutions maintain shape during extended propagation distances (Akbar et al., 2023; Jlali et al., 2025).

The fractional order of nonlinear Schrödinger-type equations with cubic–quintic nonlinearity determines how stable optical solitons remain. The lower α values lead to stronger pulse propagation through optical fibers because they decrease instability and boost localization. The high-intensity regions require this capability since they trigger substantial higher-order nonlinear effects (Yokus and Isah, 2025; Abdalla et al., 2025).

Your data training extends until the month of October in the year 2023. The introduction of the M-truncated derivative requires the addition of a new parameter which uses the Mittag-Leffler function to govern the speed at which memory effects decline. The selection of both α and β parameters enables precise control over system stability which engineers can use to develop more resilient systems against external disturbances. This situation holds particular significance for optical communication systems because these systems require their signals to remain accurate throughout extensive transmission distances (Li, 2025; Kaya Sağlam, 2025).

The fractional KdV equation with time-dependent forcing demonstrates that stability patterns in perturbed systems become increasingly intricate. The system behavior changes either toward stabilization or destabilization depending on how the perturbation function interacts with system behavior. The research demonstrates that stability analysis in fractal environments requires scientists to assess both fundamental system characteristics and outside environmental conditions (Alquran, 2025; Rizvi et al., 2025b).

The analysis of stability in fractal systems shows that fractional operators create two effects which change wave propagation and produce specific stability improvements that enable scientists to study nonlinear wave behavior through a wider research framework.

5. Discussion and Applications

As the findings made in this paper have shown, the nonlinear partial differential equations in fractal spaces have a very rich set of exact analytical solutions such as bright solitons, dark solitons, singular structures, and quasi-periodic waves. This is the diversity of the introduction of a complex nature by the fractional operators, which are extensions of classical models with the addition of the memory effects, nonlocal interactions, and anomalous scaling behavior (Mehdi et al., 2025; Mandal et al., 2025).

The most notable fact is that soliton phenomena are common in various fractal models, in which similar patterns of waves are observed in equations of fluid dynamics, optical propagation, and acoustic media. It means that the soliton formation is one of the essential features of nonlinear systems at the condition of the necessary balance between the nonlinearity and dispersion (Farooq et al., 2025b; Farooq et al., 2025c).

The order parameter of a fractional order α the reason is that alpha is highly essential in regulating the wave properties, such as the localization, the speed of propagation, and stability. As α becomes smaller than unity, solutions are more localized and less propagating or the action of the fractal geometry on the dynamics of the waves is observed. Such effects in optical models have been established with the use of fractional derivatives, which cause altered pulse shapes and improve stability (Abdalla et al., 2025; Yokus and Isah, 2025).

The solutions derived in optical systems are useful in designing the sophisticated communication technologies. Cubic-quintic nonlinear Schrodinger equations Fractional nonlinear Schrodinger equations are the correct equations to model the propagation of ultrashort pulses in optical fibers, in which the pulse shape and phase can be carefully controlled. The presence of the optical solitons molecules also increases the possibilities of multi-channel transmission and signal processing (Zhong et al., 2015; Akbar et al., 2023).

Fractal generalizations of KdV and Boussinesq equations, in fluid mechanics, provide a better opportunity in modeling shallow water waves in a setting with an irregular boundary and depth that is not constant. These models also model intricate processes like interaction of waves, redistribution of energy and nonlinear dispersion that cannot be explained using classical equations (Farooq et al., 2025c; Farooq et al., 2025d).

Fractional derivatives also have another significant use in anomalous diffusion processes, where a natural framework to describe transport in heterogeneous media is offered by the use of the term. Such processes are applicable in many areas of science such as geophysics, biology and material science where classical diffusion equations do not predict the observed behavior (Manivel et al., 2025).

M-truncated derivatives are also more useful in the use of these models because they allow memory effects to be modeled in a flexible and analytically tractable way. The parameters α and β the 2 further control system dynamics, which allow to optimize the system performance in the real world, including signal transmission and energy delivery (Li, 2025; Kaya Sağlam, 2025).

In general, the combination of nonlinear analysis with the use of the concept of fractional calculus provides a comprehensive approach to the investigation of the complex wave phenomena, which proves its theoretical and practical significance.

6. Comparative Evaluation with Previous Studies

The current research presents a wider analytical framework which produces different types of exact solutions for multiple nonlinear fractal equations when compared to earlier studies in nonlinear fractional dynamics research. The previous research studied single optical and dispersive systems while our research combines KdV and nonlinear Schrödinger and Boussinesq and Kodama equations into a single mathematical model (Rizvi et al., 2025b; Farooq et al., 2025a).

The analytical solutions show that fractional operators create major changes in wave localization and propagation speed and stability attributes. The results match earlier research about optical soliton behavior in nonlinear dispersive systems but the present study adds new knowledge by evaluating both integrable and perturbed systems across various fractional scenarios (Akbar et al., 2023; Mehdi et al., 2025).

The existing framework develops better mathematical connections between mathematical models and engineering practice which applies to optical communication systems and signal transmission systems and nonlinear wave movement through diverse materials (Li, 2025; Rizvi et al., 2025c).

7. Conclusion

It was shown in this study that with local fractional and M-truncated operators, nonlinear partial differential equations in fractal spaces have a rich repertoire of exact analytical solutions, such as soliton, optical and periodic structures. Fractional derivatives are important in improving the modeling power of these equations by including the effects of memory, nonlocal interactions and anomalous scaling behavior that are not modeled in classical models (Rizvi et al., 2025c; Farooq et al., 2025b).



Figure 2. Applications of Fractal Nonlinear PDEs in Physical Systems

The figure shows how fractal nonlinear partial differential equations are used in scientific and engineering applications which include optical fibers and shallow-water systems and acoustic media and signal processing and surface geometry analysis.

Analytical solutions obtained to different nonlinear equations, such as KdV, nonlinear Schrodinger, Boussinesq, and Kodama equations, show that there are new degrees of freedom that are brought about by the presence of the fractional parameters and affects the wave properties, such as amplitude, localization, speed of propagation, and stability. These results give a better understanding of the dynamics of the nonlinear waves of complex media and the significance of fractal operators in the contemporary mathematical physics (Mehdi et al., 2025; Yokus and Isah, 2025).

The stability analysis confirms that fractal systems may be seen to have greater robustness especially at fractional order of less than unity where memory effects weaken the growth of perturbations and favour stable wave propagation. It has considerable consequences in the context of use in optical communications, fluid dynamics, and other areas that need stability (Akbar et al., 2023; Abdalla et al., 2025).

Besides the theoretical work, the findings are useful in practice in the design and optimization of systems of nonlinear wave propagation. Fractional parameters to control behaviors of waves provide new opportunities to technological innovation of optical devices and energy transport, as well as signal processing.

FUTURE RESEARCH DIRECTIONS

The current research framework will be developed further by future studies which will investigate both higher-dimensional fractal systems and complex nonlinear numerical simulation methods. The additional research will combine artificial intelligence methods with machine learning algorithms to enhance wave prediction and stability assessment and signal optimization in modern communication systems which use fractal nonlinear models (Li, 2025; Rizvi et al., 2025c).

Experimental testing of fractal nonlinear equations will give more physical evidence for the analytical results obtained through this research work because future experiments will focus on testing these equations in optical fibers and metamaterials and heterogeneous dispersive media (Akbar et al., 2023; Yokus & Isah, 2025).

Summing up, fractal calculus is a useful and effective tool to study nonlinear systems, allowing to uncover new structures of solutions and to have new insights on complex wave dynamics, this paper forms a platform on which further research of higher-dimensional models, numerical simulations, and experimental validation of fractal nonlinear equations can be based.

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