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The Correspondence Between Links and Connective Spaces

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Abstract

Links are closely related to finite connective spaces. In this paper we explore the study of the reciprocal relationship between the finite connective spaces and links; we also characterize for the connective order of links. Finally, we show that being connective order is a connective property that is invariant under catenomorphism. Studying some properties of connective spaces by their corresponding splittability spaces is our motivation.

Keywords: Connective space; Links; Splittability structure; Irreducible set

1. Introduction

The concept of connectivity is very important in the analysis, which led G. Matheron and J. Serra (in 1988) to propose a special approach to connectivity, but in the past, this topic has not received sufficient attention. Recently, Muscat J. and Dugowson S. and others have reinforced the structural composition of these spaces, while many of their properties remain unexplored. In this paper we work in the same direction, our motivation is to explore and develop concepts specific to connective spaces. In section 2, some definitions of links and connective spaces are shown. The third section is dedicated to the splittability structures and the representation of finite connective spaces by links. The last section of the paper is devoted to the study of the connective order; also, we prove that any two catenomorphic connective spaces have the same connective order.

2. Preliminaries

In this section, we give the basic concepts. All definitions are standard and can be found in [2,8,10]

A knot is an embedding of the circle $\mathbb{S}^1$ into three-dimensional Euclidean space $\mathbb{R}^3$. A knot $\mathcal{K}$ is said to be tame if and only if it can be represented as a finite closed polygonal chain. It is wild if it is not tame. A link is a collection of disjoint knots, each of which is said to be a component of the link. In particular, a knot is a link with one component. The tame link is the link in which all components are tame, and wild otherwise. A link is called splittable if the components of the link can be deformed so that they lie on different sides of the plane in three-dimensional space. A sublink with one component is called nonsplittable. A Brunnian link is a set of n-linked loops such that each proper sublink is trivial, so that the removal of any component leaves a set of trivial unlinked .

A non-empty set X together with a collection $\mathcal{C}$ of subsets of X which satisfies the following axioms:

(i) $\emptyset \in \mathcal{C}$ and $\{x\} \in \mathcal{C}$, $\forall x \in X$

(ii) If $\{C_i : i \in I\}$ is a non-empty collection of subsets in $\mathcal{C}$ with $\bigcap_{i\in I} C_i \neq \emptyset$, then $\bigcup_{i\in I} C_i \in \mathcal{C}$.

The set X is called the support of the space $(X, \mathcal{C})$ in [4]; the collection $\mathcal{C}$ is called a c-structure of X as in [8] or a connectivity structure [2]; elements of a c-structure are called connected sets, and $(X, \mathcal{C})$ is called c-space [8] or integral connectivity space [2]. It is said to be a connective space if $\mathcal{C}$ satisfies two more conditions along with conditions (i) and (ii) as given below:

(iii) Given any non-empty sets $A, B \in \mathcal{C}$ with $A \cup B \in \mathcal{C}$, then there exists $x \in A \cup B$ such that $\{x\} \cup A \in \mathcal{C}$ and $\{x\} \cup A \in \mathcal{C}$.

(iv) If A, B, $C_i \in \mathcal{C}$ are disjoint sets and $A \cup B \bigcup_{i\in I} C_i \in \mathcal{C}$, then there exists $J \subseteq I$ such that $A \cup \bigcup_{i\in J} C_i \in \mathcal{C}$ and $B \cup \bigcup_{i\in I-J} C_i \in \mathcal{C}$

c-structure that satisfies the previous two conditions is called a connective structure or connectology on X. A connective space is called finite if the number of its points is finite. The simplest example is the discrete connective space where the discrete structure is given by $\mathcal{D} = \{\emptyset\} \cup \big\{\{x\} : x \in X\big\}$, another one is the indiscrete connective space, where the indiscrete connective structure is given by $\mathcal{I} = \mathcal{P}(X)$, The Brunnian space with n points $(B_n, \mathfrak{B}_n)$ is the space whose support X has n points and its structure $\mathfrak{B}_n = \{X, \ \{x_i\}; i = 1, \dots, n\}$. Let $\mathcal{B} \subseteq \mathcal{P}(X)$, then the intersection of all connective structures $\mathcal{C}$ on X containing $\mathcal{B}$ is a connective structure, and it is called the connective structure generated by $\mathcal{B}$ and denoted by $[\![\mathcal{B}]\!]$. A function $f: X \longrightarrow Y$ on connective spaces is called c-continuous if it maps connected sets of X to connected sets of Y. A catenomorphism is a bijection function $f: X \longrightarrow Y$ for which f and f^{-1} are c-continuous.

3. Splittability Connective Structures

Links are special examples of connective spaces. In [2], for each tame link, a connective space $(X_\ell, \mathcal{C}_\ell)$ is defined by taking the components of the link ℓ as points in X_ℓ, and with the nonsplittable sublinks of ℓ as connected subsets in $\mathcal{C}_\ell$, the structure is called the splittability structure of ℓ. As an example, the Borromean rings associate the Borromean connective spaces with three points. More generally, each Brunnian space is a connective space.

The following theorem shows that every finite connective space can be represented by a link, i.e., there exists a link whose connective structure is (isomorphic to) the one given.

Theorem 3.1. Every finite connective structure is the splittability structure of at least one link in $\mathbb{R}^3$.

The previous theorem is known as the Brunn – Debrunner – Kanenobu Theorem.

The splittability structure demonstrates the topological structure of the link and its ability to separate into disjoint parts; on the other hand, the links illustrate the complexity level of the connective structures and give a way to visualize the connections between the components of the space and more clearly show the extent of their connectivity with each other.

The following example illustrates the simplest connective space represented by a link:

Example 1. The connective space $(B_3, \mathfrak{B}_3)$, which is defined by $B_3 = \{1,2,3\}$ and $\mathfrak{B}_3 = \{\emptyset, \{1\}, \{2\}, \{3\}, B_3\}$, can be represented by Borromean link as

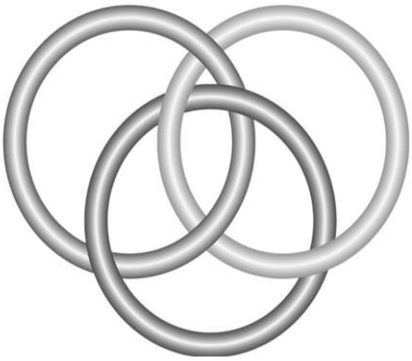

Fig 1. Borromean link

Definition 3.2. The iterated Brunnian space (Brunnian union) is the connective space $(X, \mathcal{C})$ of a non-empty family of Brunnian spaces $(X_i, \mathcal{C}_i)$, such that $X = \bigsqcup_i X_i$ (disjoint union of X_i), and $\mathcal{C} = \bigsqcup_i X_i \cup \{X\}$.

Example 2. Let $A_5 = \{1,2,3,4,5\}$, and its connective structure is

$$\mathcal{A}_5 = \begin{Bmatrix} \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \\ \{1,2\}, \{1,2,3\}, \{1,2,3,4\}, A_5 \end{Bmatrix}$$

The connective space $(A_5, \mathcal{A}_5)$ is represented by the link in Figure 2.

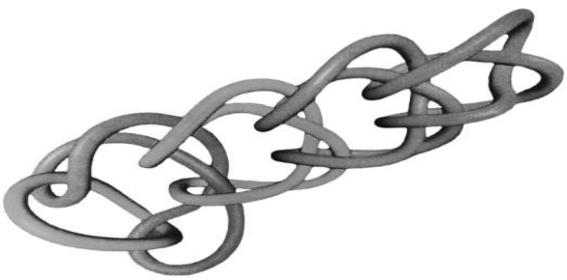

Fig 2.

Definition 3.3. [4] any pair (E, T), we called to be a collar with n components, where $T \subseteq \mathbb{R}^3$ is a solid torus and $E = (E_1, \dots, E_n)$ defines on the embedding of the link inside T such that

- E is not contained in a connected subset of T,
- For all $i \in \{1, \dots, n\}$, there exists a connected subset of T which contains $(E_j)_{j \neq i}$.

Example 3. Any completely separable link is represented by a collar; in particular, a link that consists of two non-linking circles can be represented by a collar.

Theorem 3.4. [4] The Brunnian union of a finite family of finite connective spaces that are representable by collars is itself representable by collars.

Colloray 3.5. An iterated Brunnian spaces are splittable.

Fig 3.

Example 4. The connective space $(X, \mathcal{C})$ defined by $X = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, and

$$\mathcal{C} = \begin{Bmatrix} \emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}, \\ \{8\}, \{9\}, \{1,2,3\}, \{4,5,6\}, \{7,8,9\} \end{Bmatrix}$$

We notice that each one of the sets $\{1,2,3\}, \{4,5,6\}, \{7,8,9\}$ is represented by Borromean ring, so the splittability structure can be represented by iterated Borromean rings (union Borromean of three Borromean rings), as in figure 3.

4. Order of Connective Spaces

Definition 4.1 [2] Let $(X, \mathcal{C})$ be a connective space. A connected subset R of X is said to be irreducible if $R \notin [\![\mathcal{C} \setminus \{R\}]\!]$.

Example 5. The singletons are irreducible sets in any connective space.

Our next proposition yields an alternate characterization of irreducible sets in connective spaces;

Proposition.4.2. Let $(X, \mathcal{C})$ be a finite connective space. A subset R of X is an irreducible if there are no two proper connected subsets A, B of $\mathcal{C}$, such that $R = A \cup B,\ A \cap B \neq \emptyset$.

Proof. Suppose that R is not an irreducible connected subset of X, then $R \in [\![\mathcal{C} \setminus \{R\}]\!]$, so there are two connected sets C_1 and C_2 that belong to $\mathcal{C}$, such that $R = C_1 \cup C_2$. Now from axiom (iii) of the definition of connective space, we have that there exists $x \in R$ such that $\{x\} \cup C_1 \in \mathcal{C}$ and $\{x\} \cup C_1 \in \mathcal{C}$. Choose $A = \{x\} \cup C_1 \in \mathcal{C}$, and $B = \{x\} \cup C_2 \in \mathcal{C}$, which completes the proof. □

The following proposition shows that the set of all irreducible subsets forms the minimal base for the finite connective space.

Proposition 4.3. Let $(X, \mathcal{C})$ be a finite connective space. Suppose that $\mathrm{I}(X)$ is a collection of irreducible subsets of X, then it is the minimal base for $\mathcal{C}$.

Definition 4.4. [3] Let R be an irreducible subset of finite connective space$(X, \mathcal{C})$.Then

- The order $\omega(R)$of singleton subset R is equal to zero.
- For a subset with more than one point , $\omega(R) = 1 + max_{K \in S(R)}\, \omega(K)$

Such that $S(R)$ denotes the set of all irreducible connected subsets that are strictly contained within R.

Example 6. let $X = \{1,2,3,4\}$, and $\mathcal{C} = \mathcal{D}_X \cup \left\{\{1,2\}, \{2,3\}, \{1,2,3\}, X\right\}$, then we have $\omega(\{1,2\}) = 1$, $\omega(\{2,3\}) = 1$

The connective order of irreducible subsets describes the level of interconnection between them, where the order is increasing progressively according to inclusion, starting at zero for the singletons. Higher connective order indicates a greater number of overlapping sets, which means higher complexity of the structure. The following definition illustrates the order of finite connective space:

Definition 4.5. The order $\omega(X)$ of a finite connective space $(X, \mathcal{C})$, is defined as the maximum order of its irreducible connected subsets.

Definition 4.6. The connective order of a tame link ℓ is the order of the associated connective space $(X_\ell, \mathcal{C}_\ell)$ of ℓ, i.e. $\omega(\ell) = \omega(X_\ell)$.

Example 7. Borromean space $(B_3, \mathfrak{B}_3)$ of order one, because every irreducible proper subset of it has zero order. More generally, each discrete connective space has order one.

Remark. It is noted that for each finite connective space $(X, \mathcal{C})$, since $|X| = n$, we have $\omega(X) \leq n - 1$

Definition 4.7. Let $(X, \mathcal{C})$ be an iterated Brunnian space. The order $\omega(X) = r + 1$ if it is the Brunnian union of a Brunnian space of order r and one or more other Brunnian spaces less than or equal to r.

Example 8. The link pictured in figure (2) has a connective order equal to 4**,** while the order of the link shown in figure (3) is 2**.**

Analogously to the topological space, we can define a property of a connective space that is invariant under catenomorphism as a connective property. The order of connective space is a connective property. In the next theorem, we show this.

Theorem 4.8. Two catenomorphic connective spaces have the same connective order.

Proof. Let $(X, \mathcal{C}_X), (Y, \mathcal{C}_Y)$ be two connective spaces, and $f: X \longrightarrow Y$ be catenomorphism from X to Y. Suppose that $\omega(X) = m,\ \omega(Y) = n$. For the sake of the contradiction, we assume, without loss of generality, that $m > n$. According to definition (4.5), X contains m of overlapping irreducible subsets, let's denote these sets by k_i , $i = 1, \ldots, m$ such that $k_1 \subset k_2 \subset \cdots \subset k_m$. Now consider the images of these subsets under f: $f(k_1) = R_1, f(k_2) = R_2, \ldots, f(k_m) = R_m$, since f is c-continuous and bijective, hence we have that $R_1 \subset R_2 \subset \cdots \subset R_m$. Thus $\omega(Y) = m$, contradicting the assumption. Therefore $\omega(X) = \omega(Y)$. □

In general, the converse of the last theorem is not true.

Example 9. The connective spaces of example (6) and example (7) both have the order 1, while the two spaces are not catenomorphic.

4. Conclusion

In this paper, a composition between connective structures and links has been made; finite connective spaces with links have been associated. In addition, connective order has been studied, and it is shown that it is a connective property.

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