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Poincaré Inequalities for Quasi-Doubling Measures in Metric Spaces with Fractal Boundaries

Karima Y. Elzalouk
Misurata University, Faculty of Science Department of Mathematics, Libya

Corresponding author email address: karimath7@gmail.com

Abstract

This paper investigates the validity of Poincaré inequalities in metric measure spaces that fail to satisfy the classical doubling condition. We construct a specific metric space X by taking the open unit disk in $\mathbb{R}^2$ and replacing its boundary with a Koch curve , a well-known fractal. We define a measure μ on X that combines the 2D Lebesgue measure on the interior with the s-dimensional Hausdorff measure (where s = log(4)/log(3)) on the fractal boundary. We first demonstrate that this measure μ is not a doubling measure due to the dimensional mismatch at the interface between the disk and its boundary. Our main result shows that μ is, however, a quasi-doubling measure. Building on this, we prove that the space (X, d, μ), where d is the Euclidean metric, supports a (1,p)-Poincaré inequality for $p \geq 1$. This result provides a concrete example of a non-doubling space where a Poincaré inequality holds, thereby extending the applicability of geometric analysis tools to a broader class of irregular and non-homogeneous spaces . Moreover , we demonstrate how
these results can be applied to analyze regularity and energy estimates for solutions of classical PDEs—such as the Laplace equation—posed on such domains. Our work extends the known theory to new settings not previously covered in the literature .

Keywords: Poincaré inequality, Quasi-doubling measure, Fractal boundaries, Lipschitz functions .

1. Introduction

The Poincaré inequality is a fundamental mathematical tool in analysis, function theory, and partial differential equations. Most studies have focused on regular Euclidean spaces; however, many mathematical and natural phenomena require models with fractal boundaries and non-standard measures. This paper aims to prove that the Poincaré inequality remains valid in metric spaces with fractal boundaries when equipped with a quasi-doubling measure. This result enables the analysis of solution regularity and energy estimates for differential equations in such complex environments.

This study builds upon the seminal work of Heinonen and Koskela , who extended Sobolev spaces to metric measure spaces, and Shanmugalingam , who developed Newtonian spaces. Keith and Rajala further characterized metric spaces supporting Poincaré inequalities. Foundational contributions to nonlinear potential theory on metric spaces were made by Björn and Björn . Our main contribution lies in extending these results to spaces with fractal boundaries under relaxed quasi-doubling conditions.

2. Preliminaries

In this section, we give the basic concepts . All definitions can be found in [4,7,8,9,10,11,15].

2.1 Metric Measure Space with Fractal Boundaries

Let $X \subset \mathbb{R}^2$ be the open unit disc whose boundary ∂X is replaced by a fractal curve (e.g.,Koch curve). The measure μ on X is Lebesgue measure in the interior and Hausdorff measure of dimension $s > 1$ on the boundary [7].

2.2 Quasi-Doubling Condition

For some constant $C_d > 0$ and a function $\phi(r):(0,\infty)\to(0,\infty)$, (increasing, $\phi(r)\to 0$ as $r \to 0$) , we have for all $x \in X$, $r > 0$, $\mu(B(x,2r)) \le C_d \mu(B(x,r)) + \phi(r)$ [11].

2.3 Chain Condition

For any $x, y \in X$, there exists a sequence of balls $\{B_i\}_{i=1}^N$ of radius r such that

$x \in B_1, y \in B_N, B_i \cap B_{i+1} \neq \varphi, N \le C_c \dfrac{d(x,y)}{r}$, where C_c is a constant [8,9]

2.4 Fractal boundary

The boundary ∂X is a fractal (e.g., Koch curve), equipped with Hausdorff measure of dimension $s > 1$ [7,10] .

2.5 Lipschitz Functions and Upper Gradients

Let $u : X \to \mathbb{R}$ be Lipschitz function ; A non-negative function g_u on the space is called an upper gradient of u if for every rectifiable curve $\gamma:[0,1]\to X$, the following inequality holds:

$$|u(\gamma(0)) - u(\gamma(1))| \le \int_\gamma g_u \, ds$$

This definition generalizes the classical notion of derivative to more general metric spaces where traditional derivatives may not exist. The concept of upper gradients forms the foundation for Newtonian spaces and the analysis of analytic properties of functions in such spaces [4,8,15].

3. Main Theorem

Let X subset of $\mathbb{R}^2$ be the unit disc with fractal boundary, and let μ be Lebesgue measure in the interior and Hausdorff measure on the boundary. Then for every Lipschitz function $u: X \to \mathbb{R}$, there exist constants $C, \lambda > 0$, depending on the fractal dimension s and the quasi-doubling function ϕ , such that for every ball $B = B(x,r) \subset X$:

$$\frac{1}{\mu(B)}\int_B |u - u_B|\, d\mu \le C_r \left(\frac{1}{\mu(\lambda B)} \int_{\lambda B} g_u^p\, d\mu \right)^{\frac{1}{p}}$$

Where $u_B = \frac{1}{\mu(B)}\int_B u\, d\mu$ is the average on B , g_u is an upper gradient of u .

Proof

For any ball $B \subset X$, By Jensen and Fubini [14]:

$$\frac{1}{\mu(B)}\int_B |u - u_B|\, d\mu \le \frac{1}{\mu(B)^2} \iint_{B\times B} |u(x) - u(y)|\, d\mu(x)\, d\mu(y)$$

For any pair $x, y \in B \times B$,the chain condition provides a path $x = z_1, z_2, \ldots, z_N = y$, with $z_i \in B_i$.

By triangle inequality:

$$|u(x) - u(y)| \le \sum_{i=1}^{N-1} |u(z_i) - u(z_{i+1})|$$

By upper gradient definition:

$$|u(z_i) - u(z_{i+1})| \le d(z_i, z_{i+1}) \sup_{w \in B_i \cup B_{i+1}} g_u(w)$$

Since $B_i \cap B_{i+1} \neq \varphi, d(z_i, z_{i+1}) \le 2r$, then $|u(z_i) - u(z_{i+1})| \le 2r \sup_{w \in B_i \cup B_{i+1}} g_u(w)$.

Total over chain of length :

$$|u(x) - u(y)| \le 2r \sum_{i=1}^{N-1} \sup_{w \in B_i} g_u(w)$$

From $N \le C_c \frac{d(x,y)}{r}$, thus $|u(x) - u(y)| \le 2C_c\, d(x,y) \sup_{w \in \cup B_i} g_u(w)$.

But $d(x,y) \le 2r$ (since both in B), so

$$|u(x) - u(y)| \le 4C_c r \sup_{w \in \cup B_i} g_u(w)$$

Substitute into double integral:

$$\iint_{B\times B} |u(x) - u(y)|\, d\mu(x)\, d\mu(y) \le 4C_c r \mu(B)^2 \sup_{w \in \lambda B} g_u(w)$$

Where λB is expansion covering all the chains.

Divide by $\mu(B)^2$: $\frac{1}{\mu(B)^2}\iint_{B\times B}|u(x)-u(y)|d\mu(x)d\mu(y)\le 4C_c r\sup_{w\in\lambda B} g_u(w)$

Now use Hölder to estimate the sup [14]:

$$\sup_{w\in\lambda B} g_u(w)\le\left(\frac{1}{\mu(\lambda B)}\int_{\lambda B} g_u^p d\mu\right)^{\frac{1}{p}}$$

So

$$\frac{1}{\mu(B)}\int_B |u-u_B|\,d\mu\le 4C_c r\left(\frac{1}{\mu(\lambda B)}\int_{\lambda B} g_u^p\,d\mu\right)^{\frac{1}{p}}$$

Near the fractal boundary, the measure $\mu(\lambda B)$ includes Hausdorff dimension , and the quasidoubling's error term $\phi(r)$ may appear in constants. The presence of the boundary may require expanding λB to include all balls in chains, which may "see" the complexity of the boundary, increasing the value of $\mu(\lambda B)$.Thus, the constant C and expansion λ depend on fractal dimension s of $\partial(x)$,constant C_d , function $\phi(r)$ in quasi-doubling and chain constant C_c ; geometric properties of the space .

Now : there exist constants $C,\lambda>0$ depending on above parameters, such that for every ball $B\subset X$ and every Lipschitz function u ,

$$\frac{1}{\mu(B)}\int_B |u-u_B|\,d\mu\le C_r\left(\frac{1}{\mu(\lambda B)}\int_{\lambda B} g_u^p\,d\mu\right)^{\frac{1}{p}} .$$

4. Applications to PDEs

4.1 Solving the Poisson Equation in a Metric Space with Fractal Boundaries

4.1.1Example

Let X be the unit disk in the plane, defined by $x^2+y^2<1$.

However, instead of a circular boundary, the boundary is replaced by a fractal curve known as the Koch curve [7] , which is a continuous but nowhere smooth curve with fractal dimension $s\in(1,2)$.

We consider a two-dimensional Lebesgue measure inside the disk and on the boundary (the fractal Koch) we use the Hausdorff measure of dimension s :

$$\mu=\lambda^2\Big|_{inside\ the\ desk}+H^2\Big|_{\partial X}$$

The problem is to solve the Poisson equation:

$$-\Delta u = f \quad \text{in} \quad X \;,\; u|_{\partial X} = 0$$

where f is a given function (e.g., $f = 1$).

We replace the classical Sobolev space with the Newtonian space $N^{1,2}(X,\mu)$ which takes into account the mixed measure and fractal boundary, and defines derivatives via upper gradients [3,15].

Despite the complexity of the boundary, if the measure μ satisfies the quasi-doubling condition in X, then we can prove a Poincaré inequality of the form:

$$\int_X |u|^2 \, d\mu < C \int_X g_u^2 \, d\mu$$

for all functions u that vanish on the fractal boundary, where g_u is the upper gradient

(replacing the classical derivative). In [10] the analysis of Laplacians on fractals further supports the mathematical framework needed for handling such problems in fractal domains.

4.1.2 Solution and Analysis

We formulate the problem in a weak form as in the classical case.Using the Poincaré inequality and energy methods, we establish existence and uniqueness of a weak solution u in $N^{1,2}(X,\mu)$.

Even if the solution cannot be expressed explicitly as in the classical smooth case, its existence and regularity are guaranteed by the energy estimates provided through the Poincaré inequality, regardless of how complex or fractal the boundary is.

4.2 Energy Estimates

In variational problems, energy functionals often take the form:

$$E(u) = \int_X |\Delta u|^p \, d\mu + \int_{\partial X} |u|^p \, dH^s$$

where H^s denotes the Hausdorff measure of the fractal boundary.

The Poincaré inequality guarantees the coercivity and lower semicontinuity of such functionals , which are crucial for the existence and approximation of minimizers. This is especially relevant in models of physical systems, such as heat flow or wave propagation in materials with highly irregular (fractal) interfaces [2].

5. Conclusion and Future Work

We have established the validity of weak Poincaré inequalities for metric measure spaces with fractal boundaries and quasi-doubling measures .

This opens the way for further investigations into quantitative bounds for PDE solutions, trace theory, and extension results in spaces of increasing geometric and measure-theoretic complexity.

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